

History of Mathematics Project: Learning Journeys for Kids and Others

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Website Organization

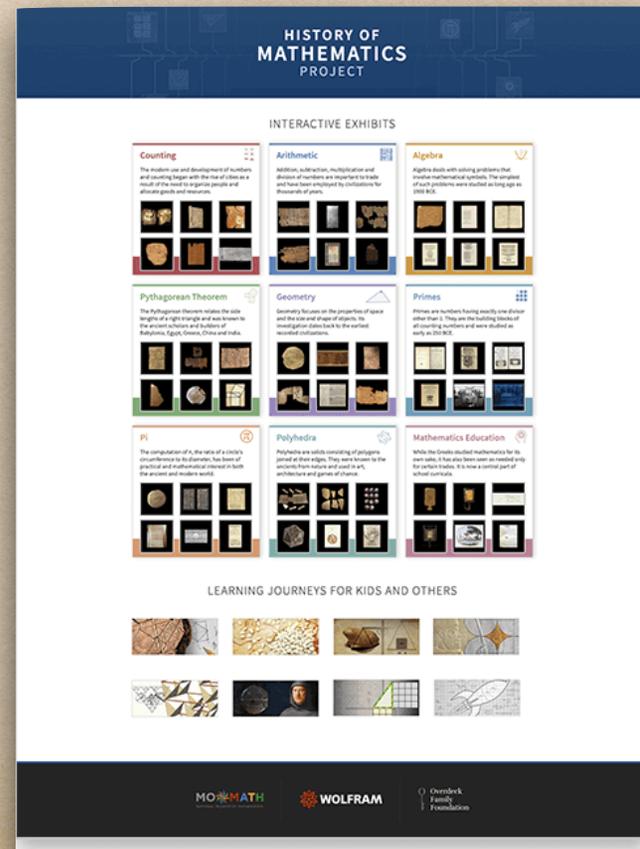
- ◆ Home page (history-of-mathematics.org)
- ◆ 9 virtual exhibits
- ◆ 74 artifact pages
- ◆ 8 learning journeys



Ancient Games of Chance

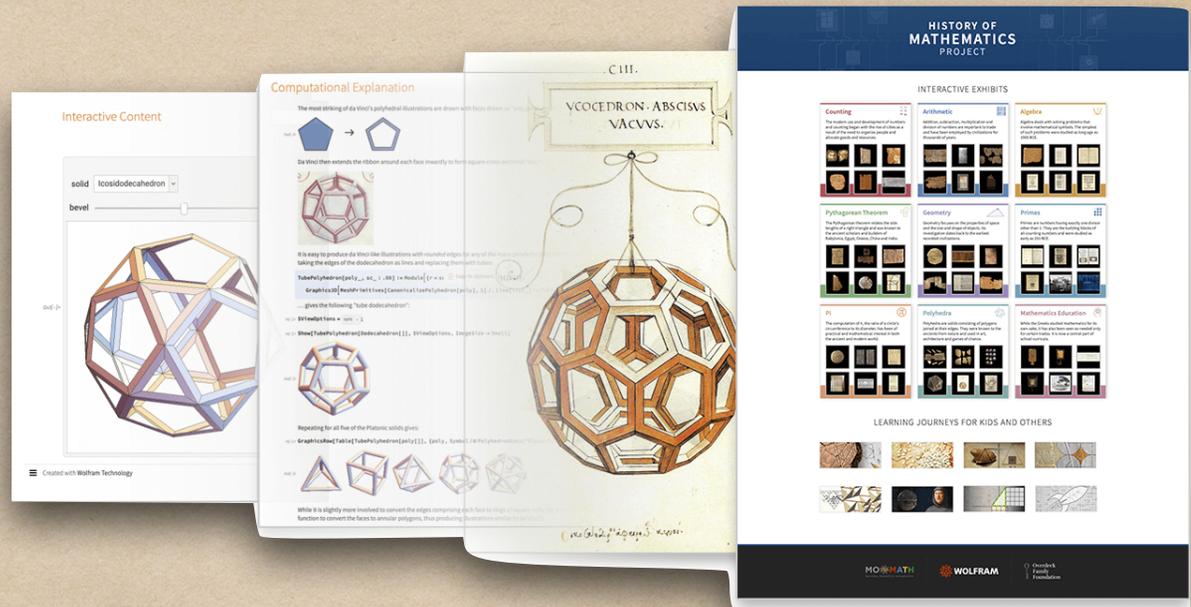


Balancing Ducks, Frogs and Grasshoppers



The History of Mathematics Development Team

- ◆ Andrea Gerlach
- ◆ Eric Weisstein
- ◆ Bernat Espigulé
- ◆ Sarah Keim Williams
- ◆ Lorí Goodman
- ◆ with additional contributions from 50+ domain experts in relevant areas of the history of mathematics, notation, and the study of antiquities



9 Interactive Exhibits + 8 Learning Journeys

Counting

The modern use and development of numbers and counting began with the rise of cities as a result of the need to organize people and allocate goods and resources.

Arithmetic

Addition, subtraction, multiplication and division of numbers are important to trade and have been employed by civilizations for thousands of years.

Algebra

Algebra deals with solving problems that involve mathematical symbols. The simplest of such problems were studied as long ago as 1900 BCE.

Pythagorean Theorem

The Pythagorean theorem relates the side lengths of a right triangle and was known to the ancient scholars and builders of Babylonia, Egypt, Greece, China and India.

Geometry

Geometry focuses on the properties of space and the size and shape of objects. Its investigation dates back to the earliest recorded civilizations.

Primes

Primes are numbers having exactly one divisor other than 1. They are the building blocks of all counting numbers and were studied as early as 250 BCE.

Pi

The computation of π , the ratio of a circle's circumference to its diameter, has been of practical and mathematical interest in both the ancient and modern world.

Polyhedra

Polyhedra are solids consisting of polygons joined at their edges. They were known to the ancients from nature and used in art, architecture and games of chance.

Mathematics Education

While the Greeks studied mathematics for its own sake, it has also been seen as needed only for certain trades. It is now a central part of school curricula.

Mathematical Beans and Knotted Strings

Balancing Ducks, Frogs and Grasshoppers

Show Your Work!

Squaring the Apsamikku Circle

Making Machines Fly

The Mathematics of a Masterpiece

Ancient Right Triangles

Ancient Games of Chance

9 Interactive Exhibits

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Mathematics Education

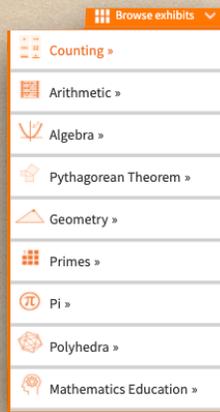
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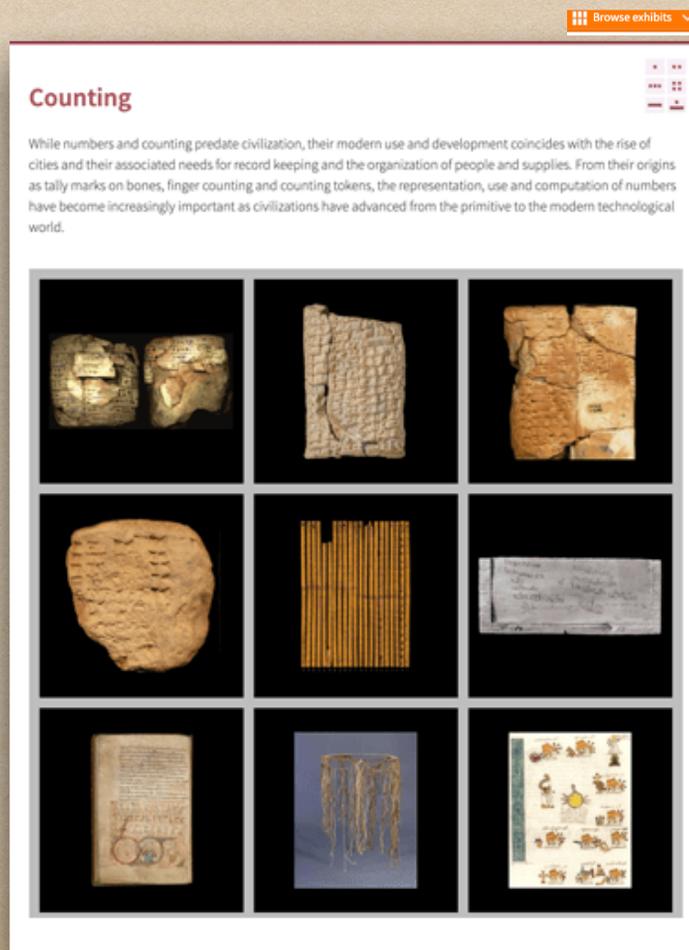
- ◆ Counting
- ◆ Arithmetic
- ◆ Algebra
- ◆ Pythagorean Theorem
- ◆ Geometry
- ◆ Primes
- ◆ Pi
- ◆ Polyhedra
- ◆ Mathematics Education

Each Interactive Exhibit Page Contains:

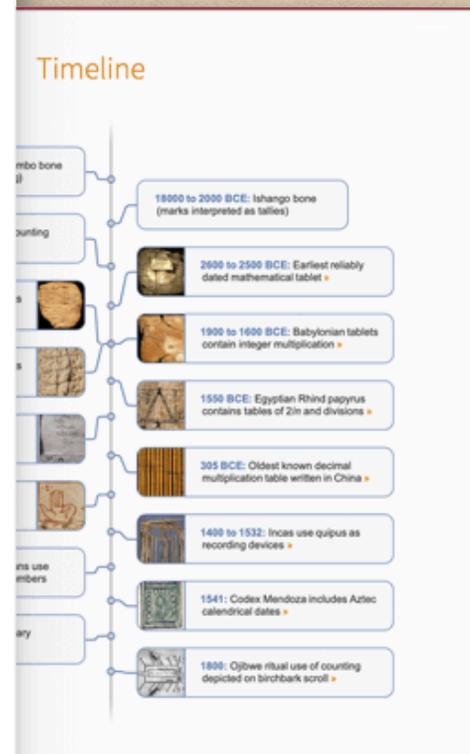
- ◆ Navigation to other exhibits



- ◆ Short description



- ◆ Clickable timeline



- ◆ Clickable thumbnails for several artifacts

Each Math Artifact Page Contains:

1615
Incan Yupana
Ancient Incan abacus

Yupanas were ancient calculating devices of the Incas. The term yupana derives from the indigenous Quechua language in which yupana means "to count." The most important historical document concerning the yupana is a sketch made by Philip Gosse in 1843, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is lost and is a 5 × 4 grid, examples depicted as polygons with different numbers of sides are also common.



The earliest known yupana has been dated to approximately 1400 CE. These early yupanas were geographically distinct from the later ones used for computation and were not used for calculation. The earliest yupana used for calculation is the one depicted by Poma de Ayala in 1552, which was a 5 × 4 grid. These early yupanas were used for computation and were not used for calculation. The earliest yupana used for calculation is the one depicted by Poma de Ayala in 1552, which was a 5 × 4 grid. These early yupanas were used for computation and were not used for calculation.

Historical sources used across the Incan Empire for storing and handling such numerical data. To perform arithmetical computations, Incan yupanas featured numbers inscribed by knots into the square using natural Yupana knots. Knots are made by pulling a string through a hole in a piece of fabric. The string is pulled through the hole in a specific way to create a knot. The string is then pulled through the hole in a specific way to create a knot. The string is then pulled through the hole in a specific way to create a knot.

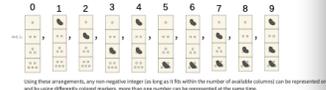
Timeline

- 1400-1500: Early yupanas used for computation.
- 1552: Poma de Ayala's 5x4 grid yupana.
- 1615: Incan Yupana artifact.
- 1700-1800: Further developments in yupana design.
- 1916: Rediscovery of the Poma de Ayala sketch.
- 2000s: Modern interpretations and digital simulations.

Interactive Content

In the above recording devices of the Andean people, knots were used to indicate numeric values based on their positions and types. In the positions of each of the squares in a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of squares which group together the tens, hundreds, thousands, tens and ones, respectively. Within each group, there are knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is calculated by adding up all the knots on the squares. Poma's (1615) and Poma's (2015) suggest the following arrangement of markers for the Incan's between 0 and 9:



Using these arrangements, we can represent the number 1000. The number of additional columns can be represented on the grid by using differently colored markers, more than one marker can represent of the same line.

Summation of multiple numbers can be performed through the mechanical operations of exchanging, moving and removing the knots on the squares. To illustrate, these operations can be tested with a starting state marked with knots together and then set to be a number. The sum can be read directly off the ending configuration of markers. This process can be viewed in a separate flow using the controls.

Historical references:
Ending state (number): 1000
Color knots: 1000s, 1000s

Starting State → Ending State

Computational Explanation

The yupanas, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua language—were the ancient calculating devices of the Incas. Several historical accounts mention the existence of these devices and describe them as a sort of tablet upon which numbers were placed. These devices were manipulated in order to perform the arithmetic calculations which would be required to solve complex problems.



Both the grids and the yupana are shown in a 1615 drawing from the 120-page manuscript *How Christian and Good Government Better Governes the Kingdom of Peru* by Pedro de Cieza de Leon. The appearance (1615) account and drawing illustrate how these devices were used while a yupana board is depicted in the lower left corner.

Starting State → Ending State

Historical Accounts

The first attempt to understand how the yupana worked was made by Henry Wessels in 1932 in an article entitled "The Ancient Peruvian Abacus," which appeared in a collection of comparative ethnographic studies edited by Oswald Neufeldt. His work was the first to attempt to reconstruct the yupana's operation based on the author's accounts of paper artifacts. The first of these accounts is the manuscript concerning Guamán Poma de Ayala's drawing.



Wessels' reconstruction, which remains the only reconstruction from which the most material, but poor quality, is that of Alexander H. Hall, the first to use the term 'yupana' to describe the device. He also suggested that the device was used for the purpose of calculating the number of soldiers in a group. The device was used for the purpose of calculating the number of soldiers in a group. The device was used for the purpose of calculating the number of soldiers in a group.

Why "Yupana" in Quechua means, corresponding to 4 × 2 × 4 = 32

Also "Yupana" in Quechua means, corresponding to 7 × 7 = 49

Also "Yupana" in Quechua means, corresponding to 4 × 4 × 16 = 256

Computing Large Sums

In order to use the yupana, the types of knots and their positions indicated numeric value. Similarly, the positions of the squares groups were manipulated to represent the numbers. The yupana board Poma de Ayala's drawing is a 5 × 4 grid of squares each having four positions (knots, thousands, hundreds, tens, and ones) and each having four positions (knots, thousands, hundreds, tens, and ones) and each having four positions (knots, thousands, hundreds, tens, and ones).

Starting State → Ending State

Back going to the first squares with 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 on them.

Interestingly, their values correspond to the Fibonacci numbers 1, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346209, 2178309, 3542248, 5710507, 9272626, 14943505, 24214301, 39186917, 63405338, 102573359, 166989687, 271853026, 438832665, 710685681, 1149432858, 1860115527, 3009548385, 4869664012, 7879202807, 12748867824, 20628066431, 33377329238, 54006186061, 87833815300, 142860383561, 232703779841, 375564163141, 608267942982, 983871722823, 1591548586804, 2575416519785, 4167018342608, 6742434862393, 10919453182001, 17661871474394, 28581324656395, 46243196130799, 74824567605193, 121016742436592, 195841360061991, 316867917717183, 512709277779174, 829571187796765, 1341280457765939, 2170851735545113, 3512132193311052, 5683083928856191, 9195215706027243, 14878067634883434, 24073281562939625, 38951297297811068, 63024578860740693, 101975866458651718, 164999145321562311, 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Each Math Artifact Page Contains:

3115
Incan Yupana
Ancient Inca objects

Yupanas were ancient calculating devices of the Inca Empire in which yupana means "to count." The idea by Felipe Guaman Poma de Ayala in 1615, which was 315. While the yupana depicted by Poma de Ayala is a 5x5 grid of dots, other versions have different shapes and sizes.

Browse exhibits

- Counting
- Arithmetic
- Algebra
- Pythagorean Theorem
- Geometry
- Primes
- Pi
- Polyhedra
- Mathematics Education

The earliest known yupana has been dated to approximately 1400 CE. These early yupanas are remarkably different from the later ones used for computation and have not been interpreted as paper boards or abacus boards. Several instruments with similar shapes were also used by Andean people. However, the type of yupana described by Poma de Ayala is 5x5 instead of a 10x10 grid which means, grids or boards were placed and manipulated to perform calculations. While there are many historical accounts of their calculating devices, not many details concerning their functioning have survived. However, it is still possible to find such about these devices were likely used.

Historical Accounts

Historical accounts of the Inca Empire for dating and handling such numerical data. To perform arithmetic computations, Inca and pre-Inca traditional numbers recorded by knots into the yupana using numerical Juyupana knots, one needs, tables or grids. It is possible that there were paper boards used for computation following a set of steps like consisting of moving and adding or like these markers until a ending state was reached.

Timeline

Timeline of Inca artifacts and documents related to the Yupana:

- 1400-1500: Inca Empire (Yupana)
- 1500-1600: Spanish Colonization (Yupana)
- 1600-1700: Colonial Period (Yupana)
- 1700-1800: Enlightenment (Yupana)
- 1800-1900: 19th Century (Yupana)
- 1900-2000: 20th Century (Yupana)
- 2000-2015: Modern Era (Yupana)

Interactive Content

In the above recording devices of the Andean people, knots were used to indicate numerical values based on their positions and types. In the positions of each of the squares in a yupana board indicated numerical value. Poma de Ayala's yupana board consists of the 25 positions of squares in which groups indicate tens, hundreds, thousands, tens of thousands, respectively. Within each group, there are knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is calculated by adding pairs, one or more numbers on the squares. Poma de Ayala's yupana board suggests the following arrangement of markers for the integers between 1 and 9:

Using these arrangements, one can represent integers up to 9 and 10 when the number of available columns can be represented on the board using differently colored markers, more than one number can be represented on the same line.

Summation of multiple numbers can be performed through the mechanical operations of exchanging, moving and restoring the knots markers on the yupana. To illustrate, these operations can be tested with a simple task: to add 12 and 15 together and store the result in the yupana. The sum can be read directly off the ending configuration of markers. This process can be viewed in a step-by-step video on the website.

Historical references:
 Ending state (yupana): 2000
 Green knots: 1000

Starting State → Ending State

Computational Explanation

The yupanas, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua language—were the ancient calculating devices of the Inca. Several historical accounts mention the existence of the device and describe it as a sort of tablet upon which numbers were placed. These devices were manipulated using beads to perform the arithmetic calculations which would be recorded onto strings from a cord.

Both the grids and the yupana are shown in a 3x3 grid from the 12th century manuscript from the University of Cambridge and the University of Cambridge. The grid is a 3x3 grid of squares. The yupana is a 3x3 grid of squares. The grid is a 3x3 grid of squares. The yupana is a 3x3 grid of squares.

Starting State → Ending State

Fibonacci (Fibonacci, 11)

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765 10946 17711 28657 46368 75025 121393 196418 317811 514229 832040 1346269 2178309 3542248 5726057 9274607 14943588 24216077 39183975 63496014 102675363 166807087 270376041 437455984 707832975 1144909064 1852391539 2997331564 4850131603 7847471177 12697602781 20547203960 33244806741 53842410722 87089617703 140912628484 227980345205 368892963009 596872610714 964785318123 1561658028838 2528384247962 4090063376001 6618871404939 10717033734903 17335805140842 28052838129781 45388822869684 73440628010623 118829440170507 192270068081130 311099410251637 503369478332767 814468539584304 1317868017616041 2132247595948745 3450015603564849 5582263201180794 9032280896739539 14612345678901234 23644626574468033 38257002471369272 61871628146837305 100119174115249277 162001802886616582 262120977001865859 424122179817115164 686143081828980953 1109264868840796122 1815406850668951286 2924671718486147408 4739878589355143694 7664540388044104982 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Each Math Artifact Page Contains:

1615
Incan Yupana
Ancient Incan abacus

Yupanas were ancient calculating devices of the Incas. The term *yupana* derives from the indigenous Andean Quechua language in which *yupana* means "to count." The most important historical document concerning the yupana is a sketch made by Felipe Guamán Poma de Ayala in 1615, which was originally lost but rediscovered at the Royal Library of Copenhagen in 1936. While the yupana depicted by Poma de Ayala is laid out in a 2 × 4 grid, examples shaped as polygons with different numbers of sides have survived.

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The earliest known yupanas have been dated to 200–600 CE. These early yupanas are morphologically distinct from the later ones used for computation and have variously been interpreted as game boards or architectural models. Musical instruments with similar shapes were also used by Andean peoples. However, the type of yupana described by Poma de Ayala in 1615 consisted of a tablet upon which stones, grains or beans were placed and manipulated to perform calculations. While there are many historical accounts of these calculating devices, not many details concerning their functioning have survived. However, it is still possible to infer much about how these devices were likely used.

Timeline

- 1615: Poma de Ayala's sketch of a yupana.
- 1936: Rediscovery of the sketch at the Royal Library of Copenhagen.
- 2000s: Archaeological excavations in Peru and Bolivia.
- 2010s: Modern reconstructions and digital simulations.

Interactive Content

In the above recording devices of the Andean peoples, knots were used to indicate numerical values based on their positions and types. For the purposes of each of the yupana based on the number value. Poma de Ayala's yupana board consists of the top grid of squares which groups indicate the thousands, hundreds, tens and ones, respectively, within each group. There are four knots representing the values 1, 2, 3 and 4.

The values being represented on a yupana (calculated by placing pebbles or more markers on the squares. Poma de Ayala's yupana (1615) and Poma de Ayala's yupana (1615) suggest the following arrangement of markers for the integers between 0 and 9:

0	1	2	3	4	5	6	7	8	9
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••

Using these arrangements, we can represent integers less than 10 if the values of the number of available columns can be represented on the grid and by using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of erasing, moving and restoring the number of markers.

Fibonacci (Poma de Ayala)

Numbers on a yupana are indicated by placing markers on the squares, where the absence of markers indicates the number zero.

The appearance of Fibonacci numbers means that the representations of 2, 3 and 5 are all made by combining the values of the two previous integers.

Therefore, the total sum obtained on the yupana is 23,100.

Starting State → Ending State

Notice that there is no need to follow a particular order since all strategies converge to the same ending state. As a result of the fact that all methods of addition take more time playing a board game than performing traditional and somewhat laborious pen and paper arithmetic computations, it is not surprising that a way of which addition could be taught to elementary mathematics classes.

A starting kit to explore and learn about addition and other arithmetical operations on the yupana known as *Yupana 1615* (Poma de Ayala) was conceived by a Peruvian team headed by the yupana expert (David Torres).



The earliest known yupanas have been dated to 200–600 CE. These early yupanas are morphologically distinct from the later ones used for computation and have variously been interpreted as game boards or architectural models. Musical instruments with similar shapes were also used by Andean peoples. However, the type of yupana described by Poma de Ayala in 1615 consisted of a tablet upon which stones, grains or beans were placed and manipulated to perform calculations. While there are many historical accounts of these calculating devices, not many details concerning their functioning have survived. However, it is still possible to infer much about how these devices were likely used.

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Historical Accounts

The first attempt to understand how the yupana worked was made by Henry Wassen in 2002 in an article entitled "The Ancient Yupana: A Re-examination of the Evidence." Wassen's interpretation is based on the written accounts of these artifacts. The first of these accounts is the manuscript concerning Guamán Poma de Ayala's drawing.

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Additional Reading

Wassen, H. "The Ancient Yupana: A Re-examination of the Evidence." *Journal of Archaeological Science*, 2002, 29(1), pp. 1-12.

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AMQP (calculadora inca)
 Con 7 figuras antropomorfas
 Incandita y ocarinas.
 Hecho de piedra y argilla cocida.
 (Sistema de un cilindro musical)
 San Pedro de Soraoca - Potosí.

Case (2012) has considered the distribution of dots into groups of 1, 2, 3 and 5 from a geometric perspective. As noted by Lovell and Shalun (2012), it has been suggested that Poma de Ayala, the inventor and author of the yupana drawing, spoke a form of Aymara. An interesting aspect of Aymara is that it uses a quinary decimal system in which 5 is used in addition to 10 when forming number words, so instead of 27 being written as "twenty seven" (not found in a quinary decimal number system), it would instead be formed as something like "twenty three and one."

To see the significance of this fact, note that the three numbers incrementally larger than 5 (which only in Aymara) on the yupana can all be formed by adding 1, 2 and 3 to the number 5 to obtain (one plus 1 = 2, 2 plus 1 = 3, 3 plus 1 = 4) and (two plus 1 = 3 and (two plus 2 = 4) and (three plus 1 = 4).

The numbers in the Aymara language therefore can be seen to coincide with the structure of yupana, making it especially accessible to Andean people speaking Aymara. As a result, the Yupana is used today as part of the curriculum in the local, bilingual schools of Poma and Bolivia.

Other Resources

South American Quipu

Additional Links

- South American Research Library and Collection: The Role of Quipu
- The Ancient Inca: A Digital Resource Center of the Royal Army, Copenhagen, Denmark
- MMA, Museo de Arte de Lima: Yupana - Inca (ca. 1400-1532) AD
- Metropolitan Museum of Art: Inca Yupana
- Museo Chileno de Arte Precolombino: Yupana and pallanes: stones to calculate
- Wikipedia: Yupana
- YouTube: Yupana Inca - Computación Andina: Una Historia de Matemáticas

Each Math Artifact Page Contains:

1615
Incan Yupana
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Browse exhibits

Interactive Content

In the above recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. Use the positions of each of the squares on a yupana board to indicate numeric value. Parts of Ayala's yupana board consists of the groups of squares which group together to be tens, hundreds, thousands, tens and units, respectively. Within each group, there are knots to represent the values 1, 2, 3 and 4.

The values being represented on a yupana (indicated by placing dots, lines or more markers on the squares). Pezaris (1992) and Pezaris (2003) suggest the following arrangement of markers for the Yupana between 0 and 9:

0	1	2	3	4	5	6	7	8	9
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
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Using these arrangements, we can represent the value 10 by using the number of squares colored red can be represented on the board by using differently colored markers, more than one marker can be represented at the same time.

Subtraction of multiple numbers can be performed through the mechanical operations of erasing, moving and restoring the number.

Fibonacci (Pezaris, 1992)

Number on a yupana are indicated by placing markers on the squares, where the absence of markers indicates the number zero.

0	1	2	3	4	5	6	7	8	9
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
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The appearance of Fibonacci numbers means that the representations of 2, 3 and 5 are all made by combining the values of the two previous squares.

1	1	2	1	2	3	2	3	5
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A PART OF AYALA'S YUPANA:

1615
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The earliest known yupanas have been dated to prior to the Inca Empire and have originated from Inca Empire or used by Andean peoples. However, the type of yupana described by Poma de Ayala and reproduced to perform calculations. While there are many examples of their functioning have combined. However, it is not

Author:
 Unknown
 Modesto Omiste, Potosí, Bolivia et al.

Current artifact location:
 Museo de Instrumentos Musicales de Bolivia, La Paz, Bolivia

Timeline

- 1615: Incan yupana first described
- 1617: Napier's Rhabdologie calculates with rods and strings
- 1640 to 1650: Printing press invented
- 1684: Pascal summarizes Renaissance mathematics
- 1693: Early slide rule developed
- 1699: Early slide rule developed
- 1700: Napier's Rhabdologie calculates with rods and strings
- 1703: Babbage difference engine
- 1740: Leibniz invents the binary system
- 1773: Gauss invents the method of least squares
- 1785: Laplace invents the method of Laplace
- 1793: Fourier invents the method of Fourier
- 1801: Babbage invents the Analytical Engine
- 1802: Fourier invents the method of Fourier
- 1804: Laplace invents the method of Laplace
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- 1899: Fourier invents the method of Fourier
- 1900: Laplace invents the method of Laplace

The earliest known yupanas have been dated to prior to the Inca Empire and have variously been used by Andean peoples. However, the type of yupana described by Poma de Ayala and reproduced to perform calculations. While there are many examples of their functioning have combined. However, it is not

Artifact format:
 Unknown

Artifact origin:
 Modesto Omiste, Potosí, Bolivia et al.

Current artifact location:
 Museo de Instrumentos Musicales de Bolivia, La Paz, Bolivia

Timeline

- 2400 to 1900 BCE: Babylonians record equations on clay tablets
- 1850 BCE: Egyptian Rhind papyrus collects mathematical problems
- 400 BCE: reckoning boards and tables used
- 330 to 31 BCE: Egyptian mathematical problem papyrus
- 100 BCE: Jain mathematicians write the Śiṣhanaga Sūtra
- 800: Al-Khwarizmi's writings include Hindu-Arabic numerals
- 1483 to 1500: Handwritten manuscripts become bound toasts
- 1693: Early slide rule developed
- 1617: Napier's Rhabdologie calculates with rods and strings
- 2000 to 1600 BCE: Egyptian scribes record mathematical problems on papyrus
- 1000 to 350 BCE: Chinese mathematical text Zhou Bi Suanjing
- 370 to 256 BCE: Eudoxus of Rhodes composes History of Arithmetic
- 300 BCE: Selenia tablet (oldest known surviving counting board)
- 300 to 1000: Bakhshali manuscript (oldest known use of zero symbol)
- 1440 to 1450: Printing press invented
- 1484: Pascal summarizes Renaissance mathematics
- 1615: Incan yupana first described
- 1619 to 1822: Babbage difference engine

to coincide with the structure of yupana, making it especially accessible to Andean people as part of the curriculum in the local, bilingual schools of Peru and Bolivia.

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Interactive and Computational Content

- ◆ Interactive content gives a Manipulate-based exploration of artifact content including some basic background and information
- ◆ Computational explanations give detailed explanations of the mathematical content of the artifact that make extensive use of the Wolfram Language

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The earliest known yupana have been dated to approximately 1400. These early yupana are morphologically distinct from the later ones used for computation and have not been interpreted as grids or abacuses. Several instruments with similar shapes were also used by Andean peoples. However, the type of yupana described by Poma de Ayala is a 5x4 grid of a table upon which stones, grains or beans were placed and manipulated to perform calculations. While there are many historical accounts of these calculating devices, not many details concerning their functioning have survived. However, it is still possible to infer much about how these devices were likely used.



Historical Orientation
Dutchman
Moderne Oosters, Petrus, Bolivia et



Museo de Instrumentos Matemáticos de Bolivia, La Paz, Bolivia

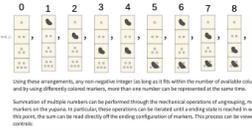
Timeline



Interactive Content

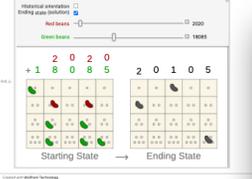
In the above recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. Similarly, the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of squares in which groups indicate ten thousands, thousands, hundreds, tens and ones, respectively. Within each group, there are four squares that represent the values 1, 2, 3 and 5.

The value being represented on a yupana is encoded by placing zero, one or more markers on the squares. Pereyra (1990) and Prem (2019) suggest the following arrangement of markers for the integers between 0 and 9:



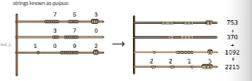
Using these arrangements, any non-negative integer (as long as it fits within the number of available columns) can be represented on the yupana and by using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of ungrouping, moving and minimizing the number of markers on the yupana. In particular, these operations can be iterated until an ending state is reached in which no further actions can be made. At this point, the sum can be read directly off the ending configuration of markers. This process can be viewed in snapshot form using the following controls:



Computational Explanation

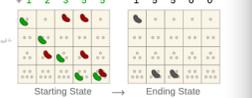
The yupana, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua calculating device of the Incas. Several historical accounts mention the existence of the device and describe its use as it has been depicted. These devices were manipulated to perform the arithmetic calculations and string between people.



Both the grids and the yupana are shown in a 360° drawing from the 1200 page manuscript from Christoph von Queneburg's historical treatise from Peru in Ayala in the European style, the apparatuses (grid) associated and quite with a yupana board depicted in the lower left corner.

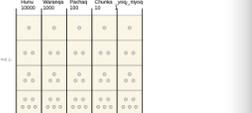


Queneburg used across the Inca Empire their starting and handling each numerical data. To perform such the yupana and to represent numbers encoded by knots into the yupana using markers. Just as the beads, the yupana enabled them to perform long computations effectively following a set of rules. The computation of these markers until a ending state was reached.



Computing Large Sums

In yupana used for record keeping, the types of knots and their positions indicated numeric value. Similarly, the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of squares (thousands, thousands (hundreds), hundreds (tens), tens (thousands) and ones (tens, hundreds, respectively).



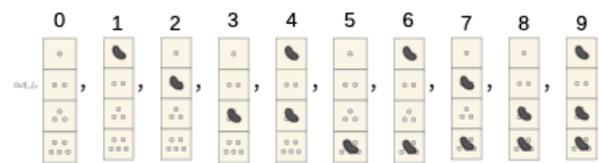
Each group has four squares with 1, 2, 3, 5 and 5, respectively, on them.

Interestingly, these values correspond to the Fibonacci numbers 1, 2, 3 and 5.

Interactive Content

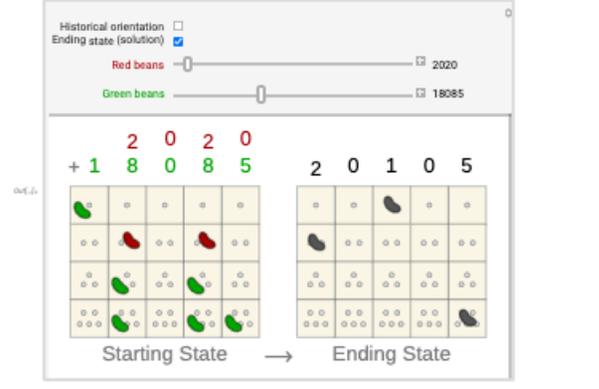
In the yupana recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. Similarly, the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of four squares in which groups indicate ten thousands, thousands, hundreds, tens and ones, respectively. Within each group, there are four squares that represent the values 1, 2, 3 and 5.

The value being represented on a yupana is encoded by placing zero, one or more markers on the squares. Pereyra (1990) and Prem (2019) suggest the following arrangement of markers for the integers between 0 and 9:



Using these arrangements, any non-negative integer (as long as it fits within the number of available columns) can be represented on the yupana, and by using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of ungrouping, moving and minimizing the number of markers on the yupana. In particular, these operations can be iterated until an ending state is reached in which no further actions can be made. At this point, the sum can be read directly off the ending configuration of markers. This process can be viewed in snapshot form using the following controls:



Created with Wolfram Technology



Historical Orientation
Dutchman
Moderne Oosters, Petrus, Bolivia et



Museo de Instrumentos Matemáticos de Bolivia, La Paz, Bolivia

Timeline



Each Math Artifact Page Contains:

1615
Incan Yupana
Ancient Incan abacus

Yupanas were ancient calculating devices of the Incas. The term yupana derives from the indigenous Andean Quechua language in which yupana means "to count." The most important historical document concerning the yupana is a sketch made by Philip Gosse in 1843, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is lost and is a 5 × 4 grid, examples depicted as polygons with different numbers of sides are also common.



The earliest known yupana has been dated to approximately 1400 CE. These early yupanas were geographically distinct from the later ones used for computation and were not used for calculation. The earliest yupana used for calculation is the one depicted in the sketch by Philip Gosse in 1843. This yupana was a 5 × 4 grid, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is lost and is a 5 × 4 grid, examples depicted as polygons with different numbers of sides are also common.

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Historical sources used around the Inca Empire for storing and handling such numerical data. To perform arithmetical computations, Inca and Spanish treasurers recorded by knots into the yupana using numerical Juyupana beads, one red, yellow or pinkish. It appears evident that the yupana had a complex structure of beads, with a set of grid lines consisting of moving and adding or subtracting these markers until a ending state was reached.

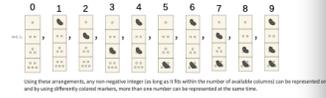
Timeline

- 1400-1500: Inca Empire (Cuzco)
- 1492: Christopher Columbus reaches the Americas
- 1532: Spanish conquest of the Incas
- 1542: Philip Gosse's sketch of a yupana
- 1916: Rediscovery of the sketch at the Royal Library of Copenhagen
- 1980s: Archaeological excavations of yupana artifacts
- 2000s: Modern reconstructions and digital simulations
- 2010s: Academic research on the mathematical structure of yupanas

Interactive Content

In the above recording devices of the Andean people, knots were used to indicate numeric values based on their positions and types. In the positions of each of the squares in a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of squares which group together the tens, hundreds, thousands, tens and ones, respectively, within each group. There are knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is indicated by placing beads, or more markers on the squares. Poma's (1602) and Poma's (2013) suggest the following arrangement of markers for the integers between 0 and 9:



Using these arrangements, we can represent integers from 0 up to 9 and the entire number of available columns can be represented on the grid using differently colored markers, more than one marker can be represented on the same line.

Simulation of multiple numbers can be performed through the mechanical operations of exchanging, moving and removing the knots markers on the yupana. In addition, these operations can be treated with a sliding block mechanism which knot together and can be used to perform. The sum can be read directly off the ending configuration of markers. This process can be viewed in a separate form using the interactive.

Historical references:
Ending state (yupana): 1000
Green beads: 10000

Starting State → Ending State

Computational Explanation

The yupana, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua language—was the ancient calculating device of the Incas. Several historical accounts mention the existence of the device and describe it as a sort of tablet upon which numbers were placed. These devices were manipulated using beads to perform the arithmetic calculations which would be required to solve complex problems.



Both the grids and the yupana are shown in a 1615 drawing from the 120th anniversary issue of the *Journal of the Royal Society*. The drawing shows the yupana board and the beads used to represent numbers. The drawing is a 5x4 grid with beads placed in the squares. The drawing is a 5x4 grid with beads placed in the squares. The drawing is a 5x4 grid with beads placed in the squares.

Historical references:
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Computing Large Sums

In order to use the yupana, the types of knots and their positions indicated numeric value. Similarly, the positions of the squares groups were manipulated through the mechanical operations of exchanging, moving and removing the knots markers on the yupana. In addition, these operations can be treated with a sliding block mechanism which knot together and can be used to perform. The sum can be read directly off the ending configuration of markers. This process can be viewed in a separate form using the interactive.

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Yupanas were ancient calculating devices of the Incas. The term yupana derives from the indigenous Andean Quechua *yupana* in which yupana means "to count." The most important historical document concerning the yupana is a sketch made by Felipe Guamán Poma de Ayala in 1615, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is lost, out is a 5 × 4 grid, exemplar shaped as polygons with different numbers of sides and other variations.



The earliest known yupana has been dated to approximately 1400 CE. These early yupanas are morphologically distinct from contemporary and later versions, have irregular or square borders or embossed circles. Several instruments with a trapezoidal shape, however, are the type of yupana described by Poma de Ayala in 1615, consisting of a table square which played and manipulated to perform calculations. While there are many historical accounts of these calculating devices, ensuring that they functioned as intended. However, it is still possible to infer their use about how these devices were used.



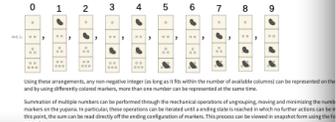
Timeline

- 1615: Felipe Guamán Poma de Ayala's sketch of a yupana.
- 1916: Rediscovery of the sketch at the Royal Library of Copenhagen.
- 1931: Hjalmar J. Hassel's reconstruction of the yupana.
- 1932: Leonard M. Shalika's work on the yupana.
- 1936: Dario E. Cuadros' work on the yupana.
- 1937: Hjalmar J. Hassel's work on the yupana.
- 1938: Hjalmar J. Hassel's work on the yupana.
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- 2019: Hjalmar J. Hassel's work on the yupana.
- 2020: Hjalmar J. Hassel's work on the yupana.
- 2021: Hjalmar J. Hassel's work on the yupana.
- 2022: Hjalmar J. Hassel's work on the yupana.

Interactive Content

In the above recording devices of the Andean peoples, knots were used to indicate numbers values based on their positions and types. Like the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of the groups of squares which groups indicate the thousands, hundreds, tens and ones, respectively. Within each group, there are knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is calculated by adding pairs, tens or more numbers on the squares. Poma's (1615) and Poma's (1916) suggest the following arrangement of markers for the integers between 0 and 9:

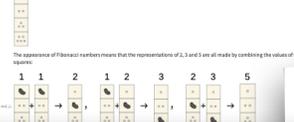


Using these arrangements, we can represent integers as long as if the values the number of available columns can be represented on the grid by using differently colored markers, more than one number can be represented on the same line.

Subtraction of multiple numbers can be performed through the mechanical operations of exchanging, moving and removing the knots markers on the yupana. In particular, these operations can be treated with a sliding knot mechanism in which the number of knots can be the same, the sum can be read directly off the ending configuration of markers. This process can be viewed in a separate flow using the controls.

Fibonacci (Poma's 1 - 1)
 (1, 1, 2, 3, 5)

Numbers on a yupana are indicated by placing markers on the squares, where the absence of markers indicates the number zero.



The appearance of Fibonacci numbers means that the representations of 2, 3 and 5 are all made by combining the values of the two previous numbers.

A PART OF A YUPANA BOARD



Other Resources

History of Mathematics
South American Quipu



Additional Reading

Acosta, J. *La Historia Natural y Moral de las Indias*. 1586. Reprinted Mexico: Fondo de Cultura Económica, 1940.

Apaza Luque, H. J. "La yupana, material manipulativo para la educación matemática." Ph.D. thesis. Madrid, Spain: Universidad Autónoma de Madrid, 2017. [link]

Ascher, M. "Mathematical Ideas of the Incas." Ch. 10 in *Native American Mathematics* (Ed. M. P. Cross). Austin, TX: University of Texas Press, pp. 261–289, esp. pp. 264–266, 1996.

Burns Glynn, W. "La tabla de cálculo de los Incas." *Boletín de Lima*, Vol. 11, pp. 57–70, 1961.

Casas, G. P. "Los Sistemas Numericos Del Quechua y el Aymara." *Revista andina*, Vol. 40, pp. 149–178, 2005.

Day, C. *Quipus and Knots: The Role of the Knot in Primitive and Ancient Cultures*. Lawrence: University of Kansas Press, 1967.

Depaulis, T. "Inca Dice and Board Games." *Board Games Studies*, Vol. 1/1998, pp. 26–49, 1998.

Leonard, M. and Shalika, C. "The Inca Abacus: A Curious Counting Device." *Journal of Mathematics and Culture*, Vol. 5, pp. 81–106, 2010. [link]

Locke, L. L. "The Ancient Peruvian Abacus." *Scripta Mathematica*, Vol. 1, pp. 36–43, 1932.

Pereira, D. "Instrumentos Prehispanicos de Calculo: el Quipu y la Yupana." *Integracion*, Vol. 4, pp. 37–56, 1986.

Pereyra, H. "El Antiguo ábaco peruano según el manuscrito de Guamán Poma." In *Quipu y Yupana*. Consejo Nacional de Ciencia y Tecnología. Lima, Perú, 1990.

Poma de Ayala, G. F. *The First New Chronicle and Good Government: On the History of the World and the Incas up to 1615* (Trans. and ed. R. Hamilton). Austin, TX: University of Texas Press, 2009.

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Tun, M. "Yupana." In *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures* (Ed. H. Selin). Dordrecht, Netherlands: Springer Science+Business Media, 2014. [link]

Valera, B. *Escol Inmaribus Blas Valera Populo Suro e Historia et Rudimento Linguae Piruorum Indios*. Napoli: Archivio Micchelli-Cera, 1618.

Wassén, H. "The Ancient Peruvian Abacus." In *Origin of the Indian Civilizations in South America* (Ed. E. Nordenskiöld). Goteborg, Sweden: Elanders Boktryckeri Aktiebolag, 1931.

Additional Links

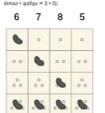
- [Dumbarton Oaks Research Library and Collection: The Role of Knots](#)
- [The Guamán Poma Website: A Digital Research Center of the Royal Library, Copenhagen, Denmark](#)
- [MALI, Museo de Arte de Lima: Yupana—Inca style 1400 AD - 1532 AD](#)
- [Metropolitan Museum of Art: Game Board \(yupana\)](#)
- [Museo Chileno de Arte Precolombino: Yupanas and pallares: Stones to calculate](#)
- [Wikipedia: Yupana](#)
- [YouTube: Yupana Inka - Competencia Método Tawa Pukllay vs Método Arábigo](#)

AMMO (calculadora Inca)
 Con 7 figuras antropomorfas
 Incandado slony y ocarinas.
 Hecho en piedra y argilla cocida.
 (Sistema de un cilindro-musical)
 San Pedro de Soreoca - Potosí.



Casa (2012) has considered the distribution of dots into groups of 1, 2, 3 and 5 from a geometric perspective. As noted by Leonard and Shalika (2010), it has been suggested that Poma de Ayala, the inventor and author of the yupana drawing, quite a bit of yupana. An interesting aspect of Poma's is that it uses a slightly different system in which 3 is used in addition to 2 when forming number words, so instead of 2 being called as "two" or "one" it would be a single digit number system, in groups, it would instead be termed as something like "two by the one two".

To see the significance of this fact, note that the three numbers incrementally larger than 3 could only be formed on the yupana can be formed by adding 1, 2 and 3 to the number to obtain (one)two + one = yupana 1 + 1 = 2, 1 + 2 = yupana 1 + 2 = 3 and (one)three + one = yupana 1 + 3 = 4.



The numbers in the Andean language therefore can be seen to coincide with the structure of yupana, making it especially accessible to Andean people speaking Yupana. As a result, the Yupana is used today as part of the curriculum in the local, bilingual schools of the area and Bolivia.

Other Resources

History of Mathematics
South American Quipu



Additional Reading

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Pereira, D. "Instrumentos Prehispanicos de Calculo: el Quipu y la Yupana." *Integracion*, Vol. 4, pp. 37–56, 1986.

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Additional Links

- [Dumbarton Oaks Research Library and Collection: The Role of Knots](#)
- [The Guamán Poma Website: A Digital Research Center of the Royal Library, Copenhagen, Denmark](#)
- [MALI, Museo de Arte de Lima: Yupana—Inca style 1400 AD - 1532 AD](#)
- [Metropolitan Museum of Art: Game Board \(yupana\)](#)
- [Museo Chileno de Arte Precolombino: Yupanas and pallares: Stones to calculate](#)
- [Wikipedia: Yupana](#)
- [YouTube: Yupana Inka - Competencia Método Tawa Pukllay vs Método Arábigo](#)

8 Learning Journeys

Counting

The modern use and development of numbers and counting began with the rise of cities as a result of the need to organize people and allocate goods and resources.



Arithmetic

Addition, subtraction, multiplication and division of numbers are important to trade and have been employed by civilizations for thousands of years.



First cord 1989
 Second cord 2021
 Third cord 1342

+

1	9	8	9	1989
2	0	2	1	2021
1	3	4	2	1342
5	3	5	2	5352

Historical orientation
 Ending state (solution)

Red beans 2020
 Green beans 1805

2 0 2 0
 + 1 8 0 8 5

2	0	2	0
1	8	0	8
0	0	0	0
0	0	0	0

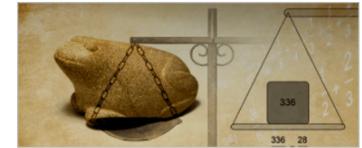
Starting State →

2	0	1	0	5
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Ending State



Mathematical Beans and Knotted Strings



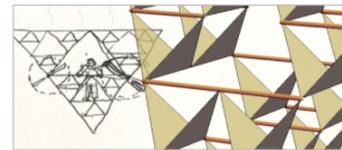
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



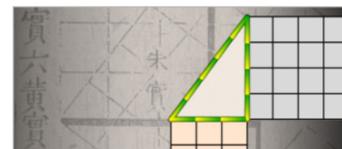
Squaring the Apsamikku Circle



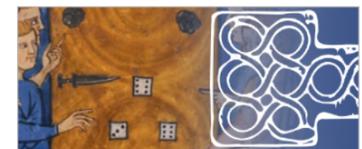
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles

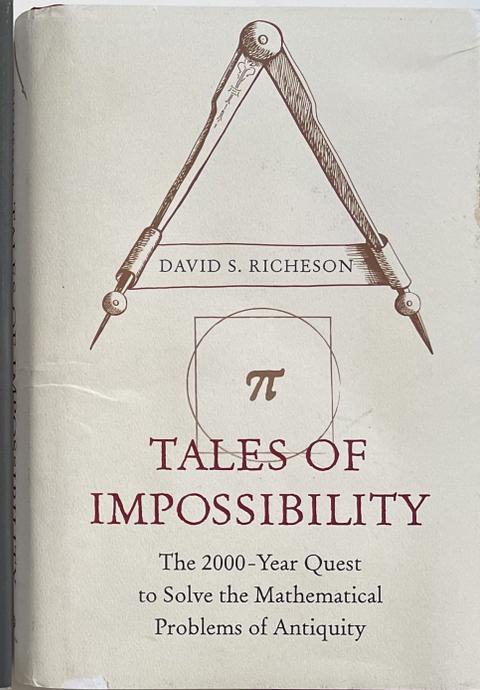
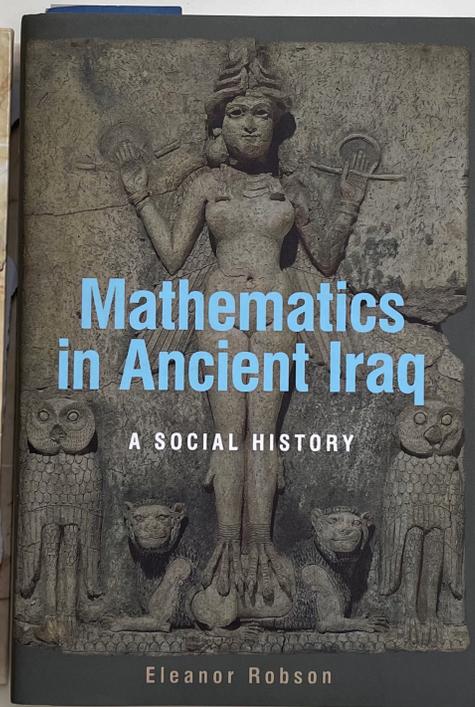
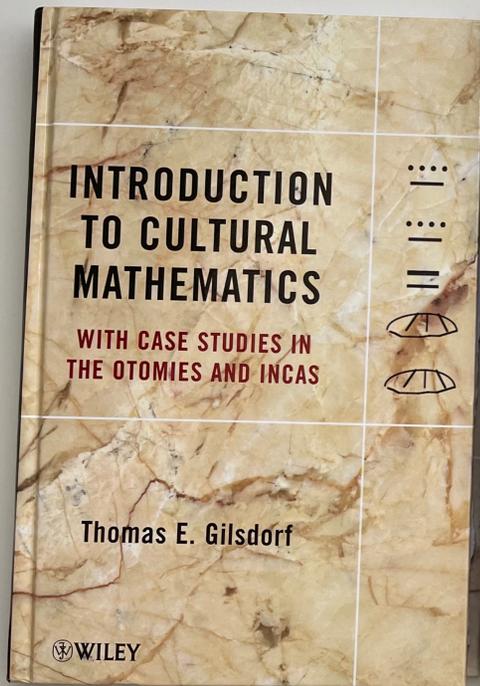


Ancient Games of Chance

Learning Journeys

- ◆ Intended as an engaging and fun "journey" through mathematical artifacts
- ◆ Aimed at students and other virtual museum visitors who are interested in the "mathematical story"
- ◆ Useful for classroom exploration or as a teaching tool
- ◆ Contain images and links to individual artifacts
- ◆ Include interactive content
- ◆ Primarily visual and descriptive with minimal mathematics
- ◆ 8 learning journeys

Learning Journeys

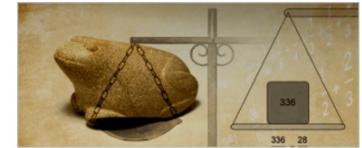


8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
- ◆ Balancing Ducks, Frogs and Grasshoppers
- ◆ Show Your Work!
- ◆ Squaring the Apsamikku Circle
- ◆ Making Machines Fly
- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
- ◆ Ancient Games of Chance



Mathematical Beans and Knotted Strings



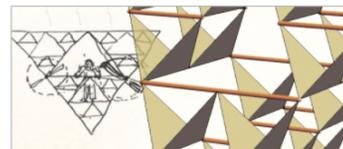
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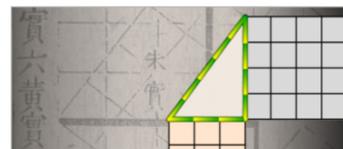
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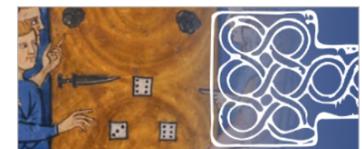
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles



Ancient Games of Chance

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Mathematical Beans and Knotted Strings



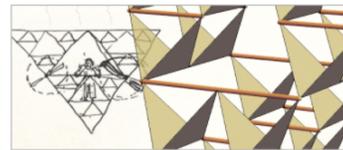
Balancing Ducks, Frogs and Grasshoppers



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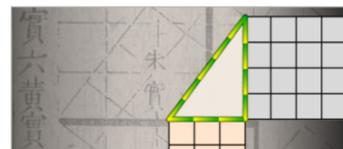
Squaring the Apsamikku Circle



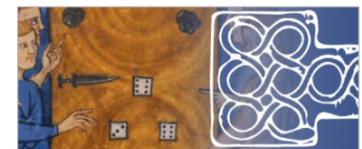
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles



Ancient Games of Chance

Ancient Games of Chance

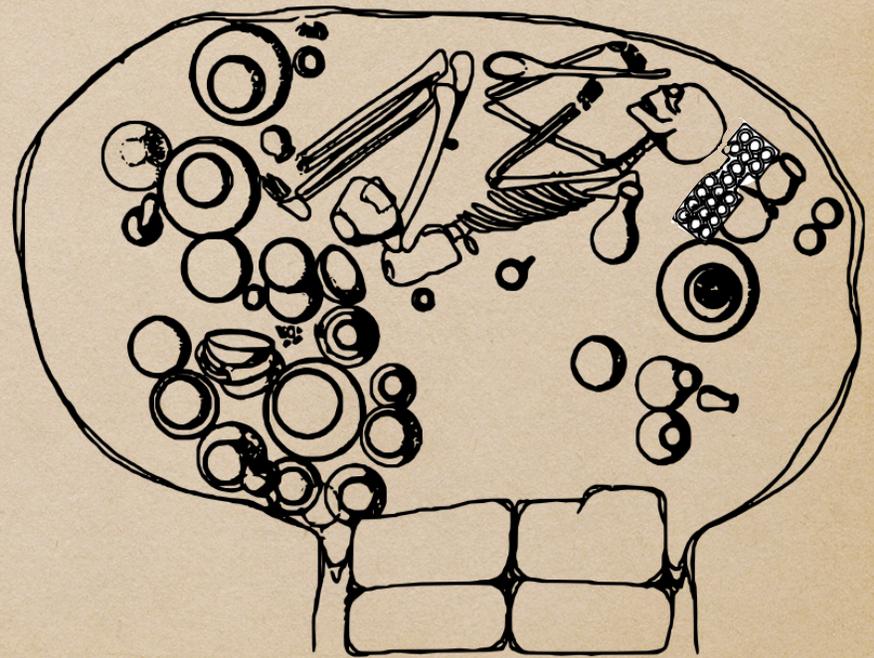
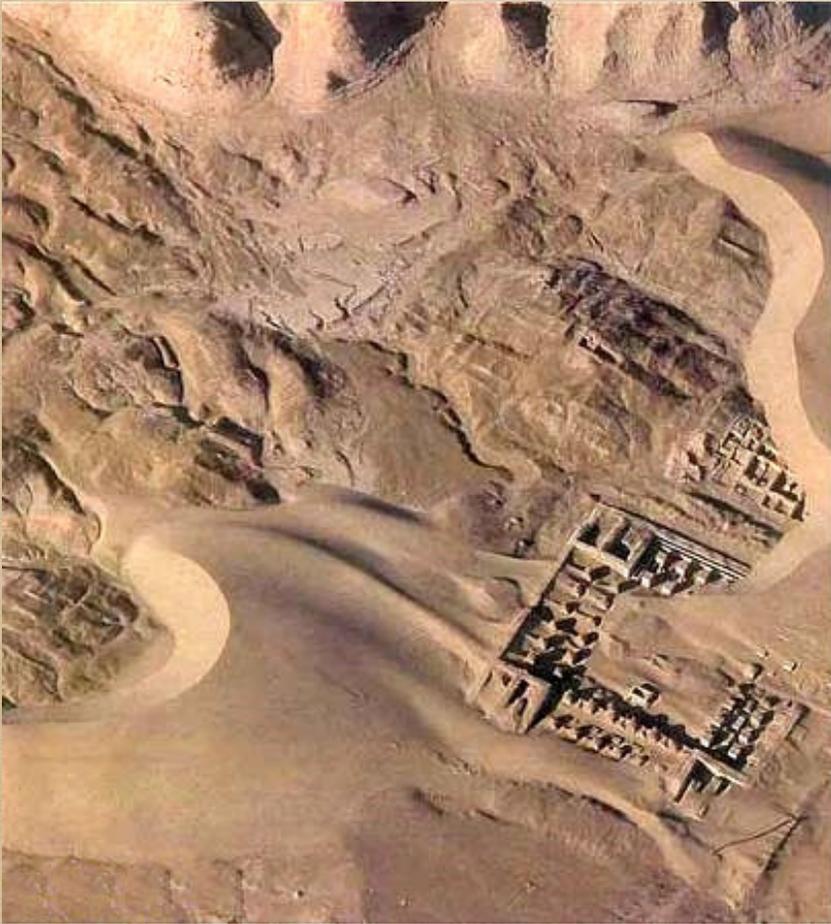
Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



Dice crafted about 4500 years ago and discovered in the 1970s by an Italian expedition to the Burnt City ruins located in nowadays Iran, midway between the Middle East and the Indus Valley, India.

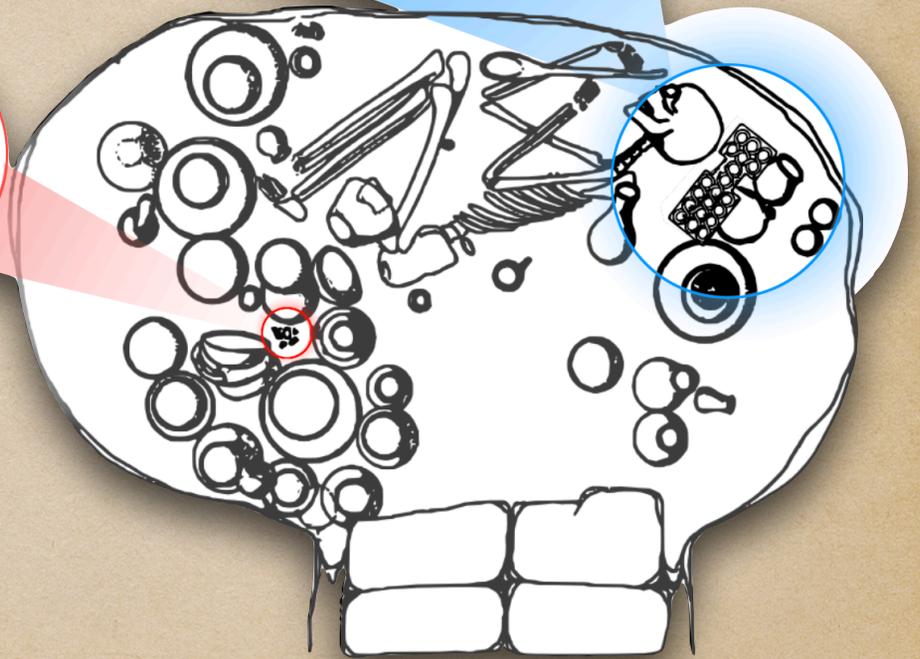
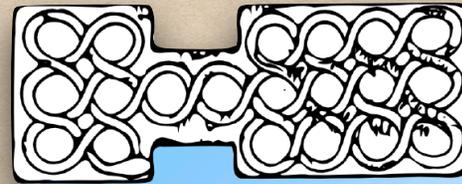
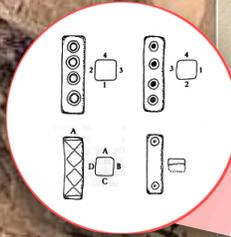
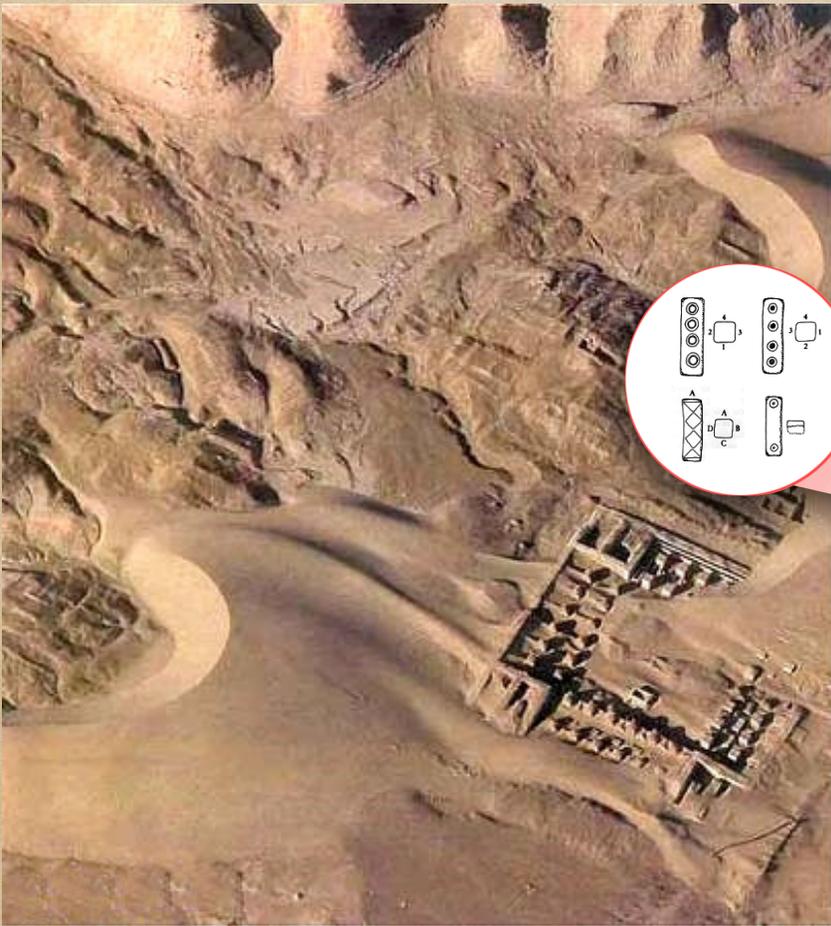
Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")

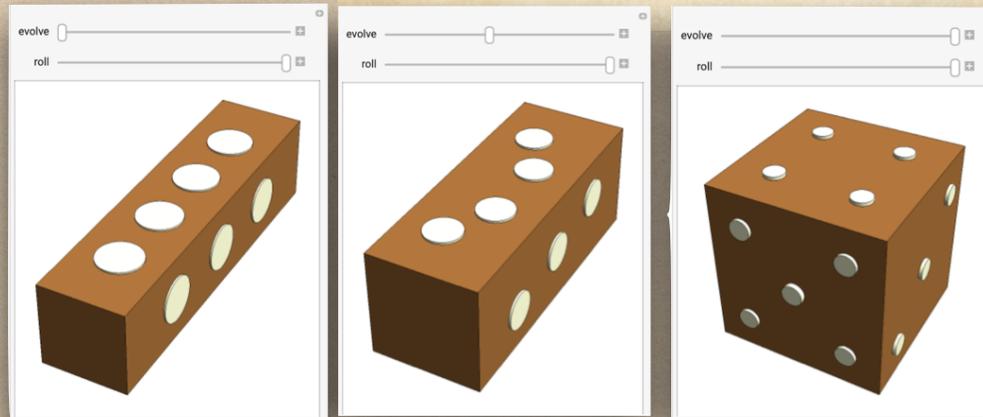
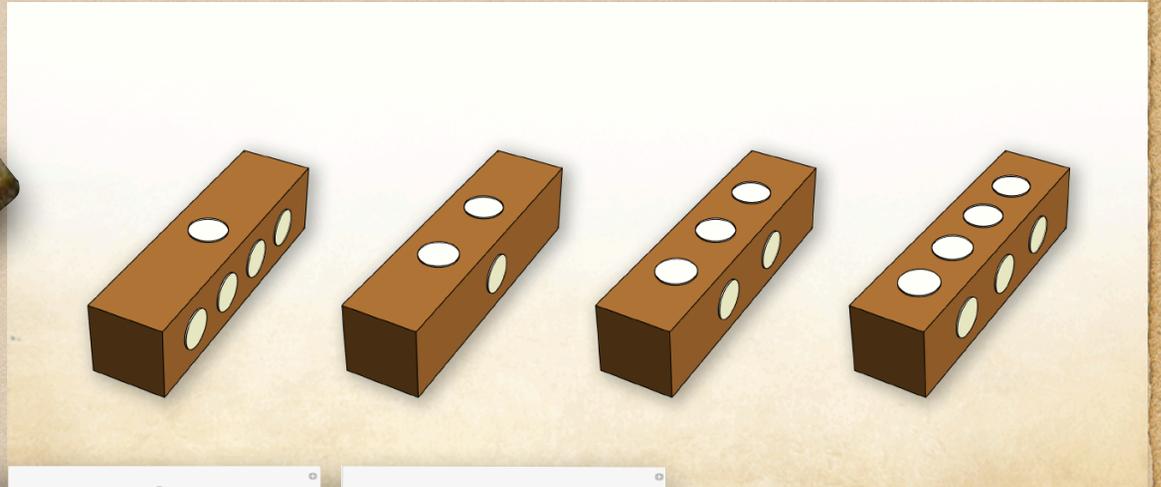


Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



Ancient Games of Chance



Four-valued die from the Burnt City
morphed to a modern six-sided cubical die.

Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



The game board was adorned with a knotted snake carved in relief and is identical in layout to a board found in the Royal Cemetery at Ur.

Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



Game board from the Royal Cemetery at Ur (British Museum item #120840), which has been dated to 2600 BCE. While the Ur board is identical in form to the Burnt City board, it is much more regally adorned, including many beautiful and intricate geometric patterns.

Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



While the Game of Twenty Squares is the world's oldest known board game, it can still be played today, since its rules have been deciphered by Dr. Irving Finkel, assistant keeper of ancient Mesopotamian script, languages and cultures in the Middle East department at the British Museum.

Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.

Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.

White stone die. 30 BCE–364 CE. Roman period.

Ancient Games of Chance

The process of making fair dice from Folio 65v of the hand-illuminated manuscript the *Book of Games*, or *Libro de axedrez, dados e tablas* (Book of Chess, Dice and Tables, in Old Spanish).



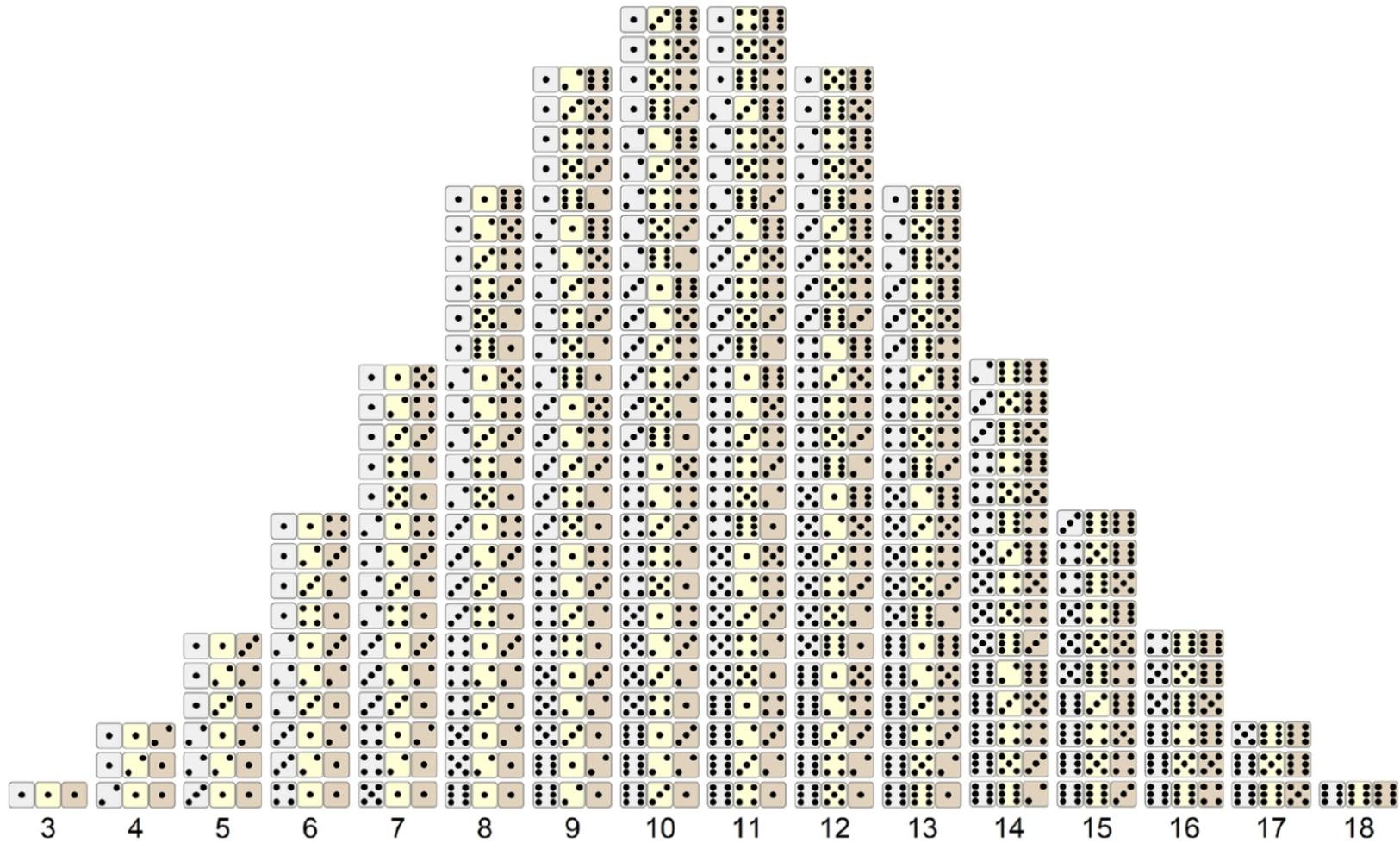
During the Greco-Roman period, cubic dice became more common and gradually replaced throwing sticks and knucklebones for use with board games.

Ancient Games of Chance

Two winning triga rolls. Libro de axedrez, dados e tablas, Fol. 66r.



Ancient Games of Chance



The probabilities for obtaining a given total using three dice, which approaches a *normal distribution*.

Ancient Games of Chance

Galileo stated that with three dice, there can only be one way of obtaining a 3 (1, 1, 1) and an 18 (6, 6, 6). However, there are three combinations for obtaining a 6—(4, 2, 1), (3, 2, 1) and (2, 2, 2)—which can occur in different orders, making 10 possibilities. There are four combinations for a 7—(5, 1, 1), (4, 2, 1), (3, 3, 1) and (3, 2, 2)—which lead to 15 possibilities. However, although 9 and 12 could be made up in the same number of ways as 10 and 11, from their experience, gamblers claimed that the occurrence of 10 and 11 were more likely! Galileo showed that the total number of possible throws with three dice are 216, and he gave a table of the number of possible throws for a total of 10, 9, 8, 7, 6, 5, 4 and 3, showing that the throws for 11 to 18 were symmetrical with these. In this way, he showed that there were 27 possible throws to obtain a 10, and 25 for a 9.

8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
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- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
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Mathematical Beans and Knotted Strings



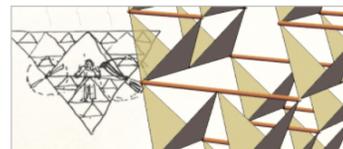
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



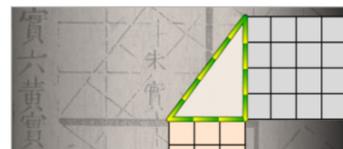
Squaring the Apsamikku Circle



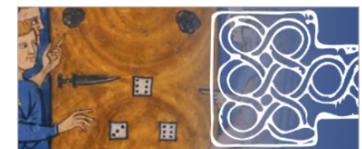
Making Machines Fly



The Mathematics of a Masterpiece



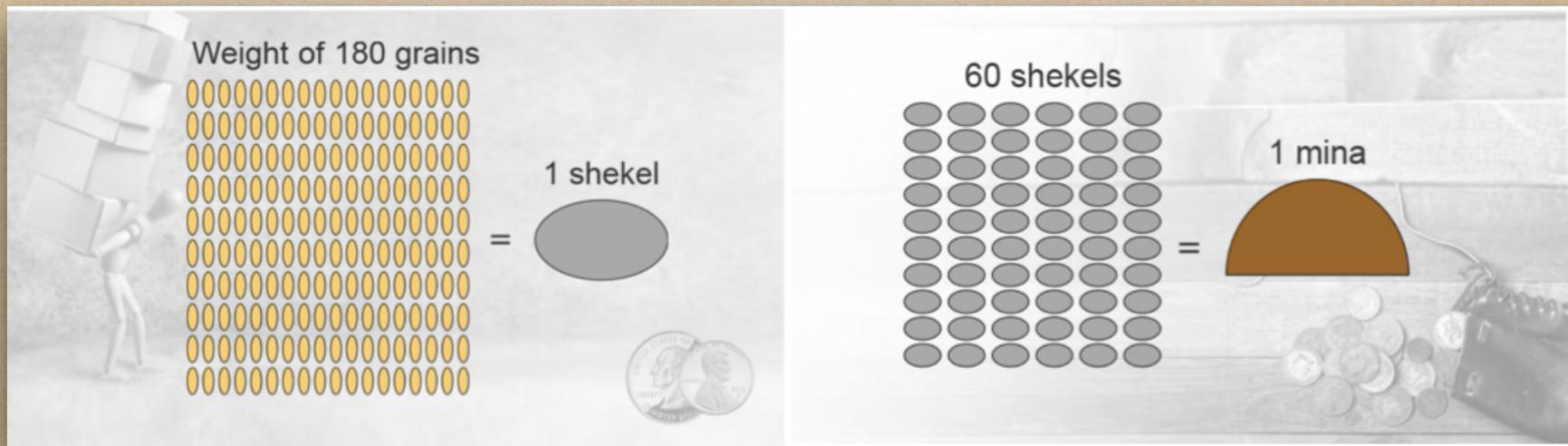
Ancient Right Triangles



Ancient Games of Chance

Balancing Ducks, Frogs and Grasshoppers

Barley was so important to the ancient Mesopotamians that a barley grain was used as the smallest unit of length, area, volume and weight. A shekel of silver weighed as much as 180 barley grains, or about 8.4 grams. 60 shekels weighed 1 mina, and 60 mina weighed 1 talent.



Balancing Ducks, Frogs and Grasshoppers

Merchants would carry around their own set of weights to help them with trading. Most weights were sort of grain-shaped. Mesopotamian weights were often made of polished **hematite**:



Hematite weights ranging from three shekels to one mina. Uruk, Mesopotamia, ca. 2000–1600 BCE.

Mesopotamian weights were often shaped like a sleeping duck, with its neck and head resting on its back:



Left: Duck-shaped hematite weights, Mesopotamia, ca. 2000 BCE.
Right: A sleeping duck!

There are rare examples of Mesopotamian weights in other shapes, but most weights were either grain-shaped or duck-shaped. Here are some unusual examples of Mesopotamian weights: a grasshopper, a shell and a cute frog:



Left: Mesopotamian grasshopper weight made of hematite, ca. 1800–1600 BCE.
Center: Mesopotamian shell weight made of hematite, ca. 1800–1600 BCE.
Right: Mesopotamian frog weight, ca. 2000–1600 BCE. The Akkadian inscription under the frog's throat reads: "a frog [weighing] 10 mina, a legitimate weight of the god Shamash, belonging to Iddin-Nergal, son of Arkat-ili-damqa."

Balancing Ducks, Frogs and Grasshoppers

- ◆ To understand how these weight stones might have been used I created the following balance scale interactive. As you add more barley grains on the left side of the scale, the merchant adds duck weights that come in fractions of a shekel so the sides balance. The beam at the top of the scale acts as an "equal" sign!

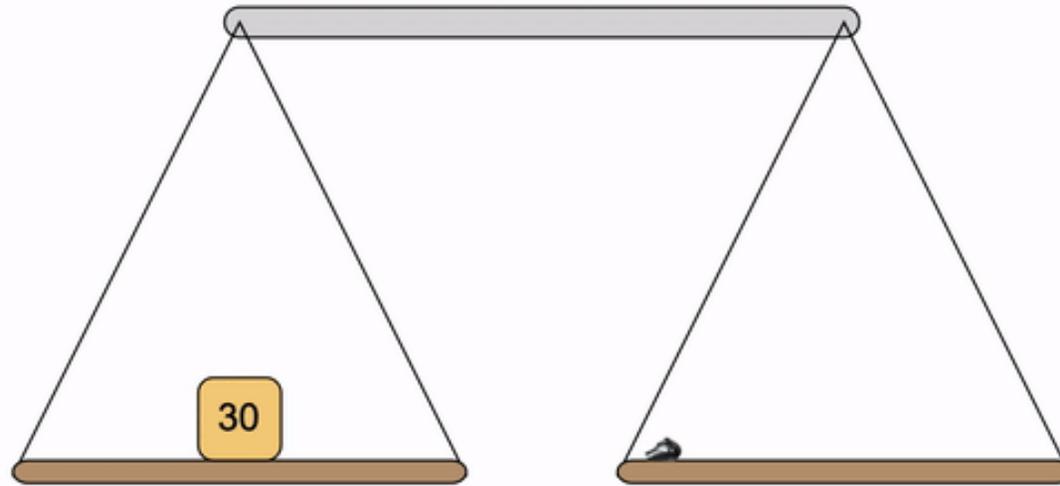
Bala

Weight units: Grams Ounces

Barely grains:

1.4 g

$$\frac{1}{6} = \frac{1}{6}$$



$$\frac{30}{180} = \frac{1}{6} \text{ shekel}$$

$$\frac{1}{6}$$

- ◆ To have scale graduated she

top of the scale acts as an "equal" sign!

pers

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nce

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e

Benefits of a Wolfram Language build system

- ◆ Easy to curate all relevant data (text, images, metadata, mathematical and interactive content) in a single place: notebooks
- ◆ Content elements such as maps, timelines, and thumbnails can be generated completely programmatically using built-in Wolfram Language functionality
- ◆ Incremental builds are easy simply by checking for changed notebook content

Thanks!

- ◆ Overdeck Family Foundation
- ◆ MoMath, the Museum of Mathematics in New York City.
- ◆ Stephen Wolfram
- ◆ Andrea Gerlach, Eric Weisstein, Sarah Keim Williams
- ◆ Lori Goodman, Sushma Kini (project management)
- ◆ Michael Trott (content suggestions and review), Christopher Wolfram (content suggestions and discussions), Dan McDonald (synthetic geometry contributions), MinHsuan Peng (custom timelines), Shadi Ashnai and Giulio Alessandrini (image processing)
- ◆ Heidi Kellner and Jeremy Davis (web design), Marion Morris (web implementation), Taylor Birch (proofreading)
- ◆ Our network of 50+ domain and content experts

Background and Timeline

- ◆ In 2019, Stephen Wolfram proposed a project to develop a virtual interactive collection of mathematical artifacts for the Museum of Mathematics (MoMath) in New York City
- ◆ The project was generously funded by Overdeck Family Foundation
- ◆ Over the last two years, researchers at Wolfram Research have investigated and written up detailed histories, descriptions, and explanations for a collection of mathematical artifacts
- ◆ The results have been incorporated into a website (history-of-mathematics.org) created using a custom build system modeled after the one being used for Stephen Wolfram's Physics Project

Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]
- ◆ All content built to and hosted in the Wolfram Cloud
- ◆ Computational/interactive content are simply notebook sections embedded directly in the cloud using WolframNotebookEmbedder
- ◆ Core workflow based on XMLTemplate + ExportForm:

Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]

- ◆

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  Permissions → "Public"]
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- ◆ Core workflow based on XMLTemplate + ExportForm:

8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings. Counting Methods from the Moche Culture.
- ◆ Balancing Ducks, Frogs and Grasshoppers. Weights and Measures in Ancient Mesopotamia.
- ◆ Show Your Work! Doing Math Homework on Clay Tablets, Papyri, Wax Tablets, Bamboo Strips and Birch Bark.
- ◆ Squaring the Apsamikku Circle. The Search to Solve One of the Oldest Problems in Math.
- ◆ Making Machines Fly. Overcoming the Square-Cube Law.
- ◆ The Mathematics of a Masterpiece. Portrait of Luca Pacioli.
- ◆ Ancient Right Triangles. The Pythagorean Theorem and the Gou-Gu Rule.
- ◆ Ancient Games of Chance. The Beginnings of the Mathematical Theory of Probability.