

History of Mathematics Project: Learning Journeys for Kids and Others

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   #WolframTechConf

@bernatree

Website Organization

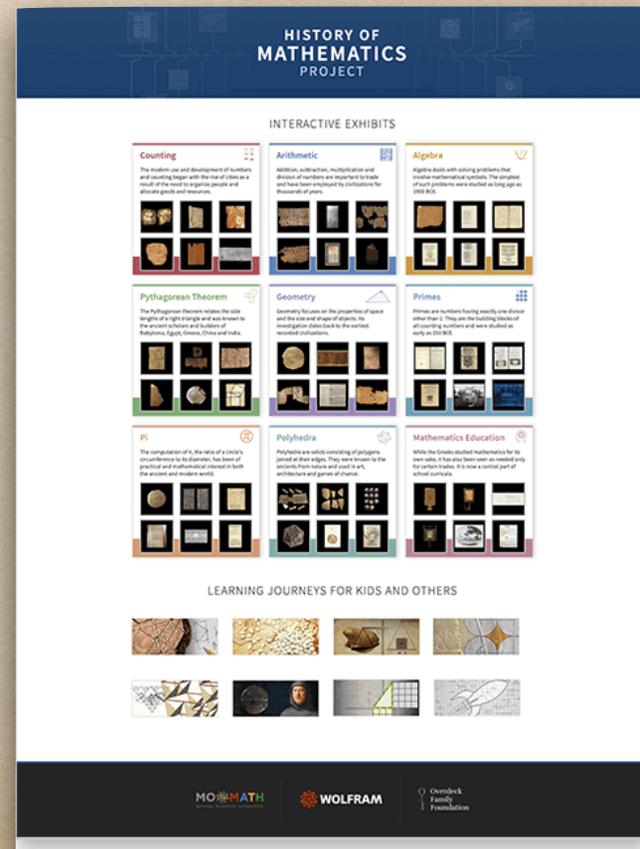
- ◆ Home page (history-of-mathematics.org)
- ◆ 9 virtual exhibits
- ◆ 74 artifact pages
- ◆ 8 learning journeys



Ancient Games of Chance

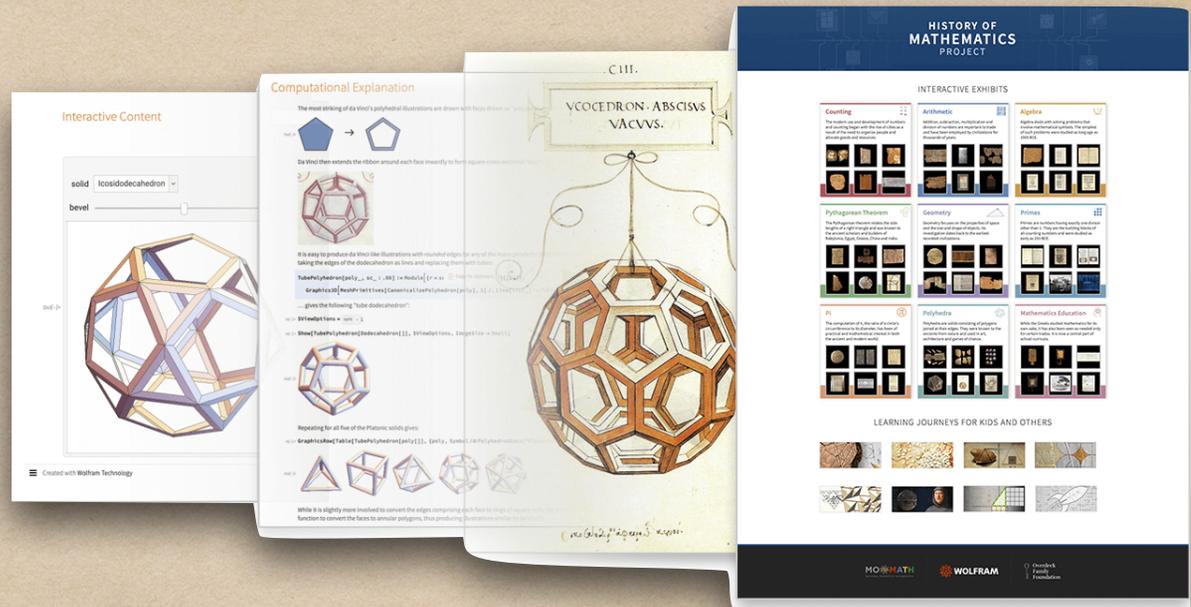


Balancing Ducks, Frogs and Grasshoppers



The History of Mathematics Development Team

- ◆ Andrea Gerlach
- ◆ Eric Weisstein
- ◆ Bernat Espigulé
- ◆ Sarah Keim Williams
- ◆ Lorí Goodman
- ◆ with additional contributions from 50+ domain experts in relevant areas of the history of mathematics, notation, and the study of antiquities



9 Interactive Exhibits + 8 Learning Journeys

Counting

The modern use and development of numbers and counting began with the rise of cities as a result of the need to organize people and allocate goods and resources.

Arithmetic

Addition, subtraction, multiplication and division of numbers are important to trade and have been employed by civilizations for thousands of years.

Algebra

Algebra deals with solving problems that involve mathematical symbols. The simplest of such problems were studied as long ago as 1900 BCE.

Pythagorean Theorem

The Pythagorean theorem relates the side lengths of a right triangle and was known to the ancient scholars and builders of Babylonia, Egypt, Greece, China and India.

Geometry

Geometry focuses on the properties of space and the size and shape of objects. Its investigation dates back to the earliest recorded civilizations.

Primes

Primes are numbers having exactly one divisor other than 1. They are the building blocks of all counting numbers and were studied as early as 250 BCE.

Pi

The computation of π , the ratio of a circle's circumference to its diameter, has been of practical and mathematical interest in both the ancient and modern world.

Polyhedra

Polyhedra are solids consisting of polygons joined at their edges. They were known to the ancients from nature and used in art, architecture and games of chance.

Mathematics Education

While the Greeks studied mathematics for its own sake, it has also been seen as needed only for certain trades. It is now a central part of school curricula.

Mathematical Beans and Knotted Strings

Balancing Ducks, Frogs and Grasshoppers

Show Your Work!

Squaring the Apsamikku Circle

Making Machines Fly

The Mathematics of a Masterpiece

Ancient Right Triangles

Ancient Games of Chance

9 Interactive Exhibits

Counting

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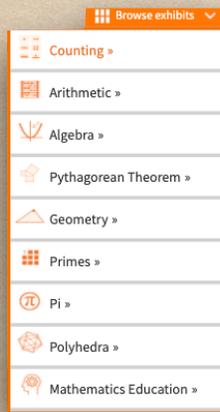
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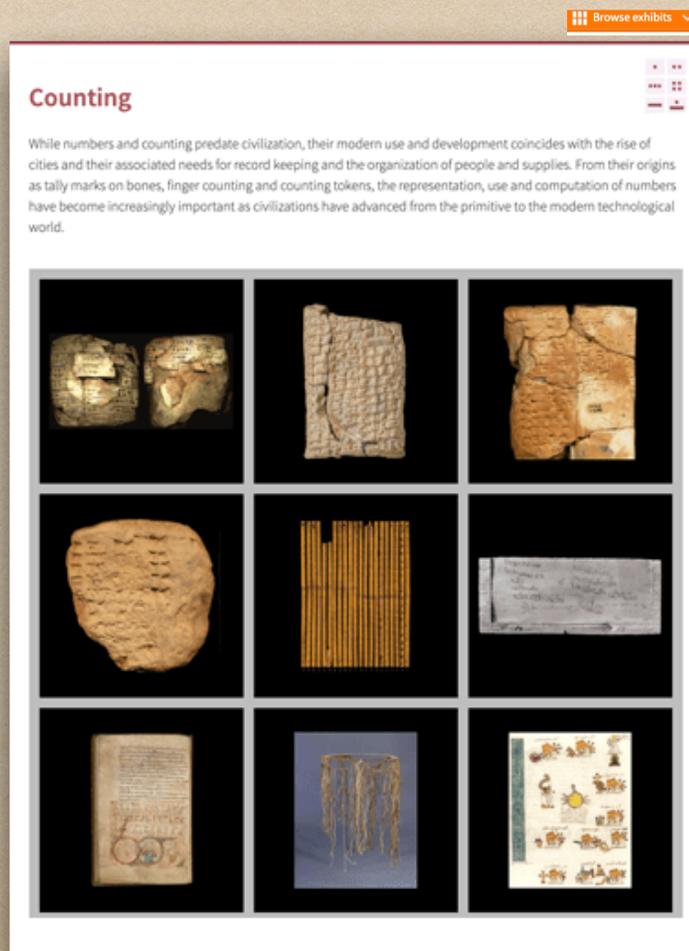
- ◆ Counting
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- ◆ Mathematics Education

Each Interactive Exhibit Page Contains:

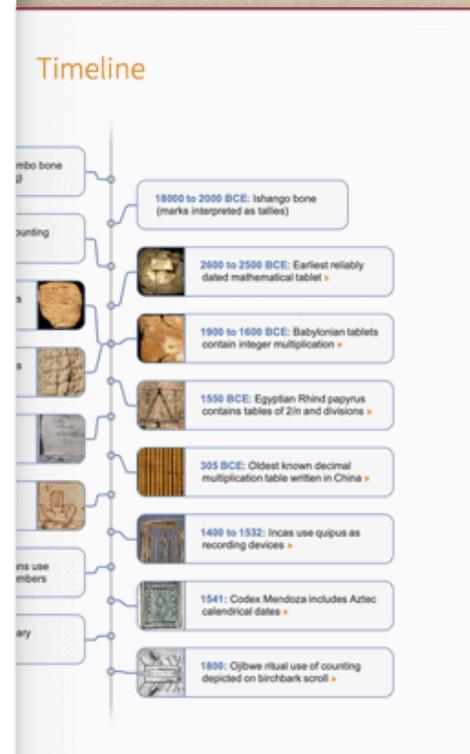
- ◆ Navigation to other exhibits



- ◆ Short description



- ◆ Clickable timeline



- ◆ Clickable thumbnails for several artifacts

Each Math Artifact Page Contains:

1615
Incan Yupana
Ancient Incan abacus

Yupanas were ancient calculating devices of the Incas. The term yupana derives from the indigenous Andean Quechua language in which yupana means "to count." The most important historical document concerning the yupana is a sketch made by Felipe Guaman Poma de Ayala in 1615, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is laid out in a 5 × 4 grid, examples shaped as polygons with different

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The earliest known yupana have been dated to prior to 1615. These early yupana are morphologically distinct from the later ones used for computation and have been found in various locations in the Andean region. Several yupana with similar shapes were also used by Andean peoples. However, the types of yupana described by Poma de Ayala in 1615 consisted of a solid square which means, grids or bases were placed and manipulated to perform calculations. While there are many historical accounts of these calculating devices, not many details concerning their functioning have survived. However, it is still possible to infer such details from these devices were likely used.

Timeline

- 1615: Felipe Guaman Poma de Ayala's sketch of a yupana.
- 1916: Rediscovery of the sketch at the Royal Library of Copenhagen.
- 1920s-1930s: Early archaeological findings of yupana.
- 1950s-1960s: Further archaeological discoveries and studies.
- 1970s-1980s: Academic research and publications on the topic.
- 1990s-2000s: Continued archaeological work and historical analysis.
- 2010s-Present: Modern educational and digital artifacts.

Timeline

Interactive Content

In the above recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. So the positions of each of the squares in a yupana board indicated numeric value. Poma de Ayala's yupana board consists of the groups of squares which group together the tens, hundreds, thousands, tens and ones, respectively, within each group. There are four knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is indicated by placing one, two or more markers on the squares. Poma's (1615) and Poma's (1916) suggest the following arrangement of markers for the integers between 1 and 9:

Using these arrangements, we can represent integers less than 10 by the value the number of available columns can be represented on the grid and using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of counting, moving and restoring the number.

Both the grids and the yupana are shown in a 1615 sketch from the 1615 manuscript from Christian and David Guaman Poma's sketch of the Quechua individual that Guaman Poma de Ayala in the illustration, the appearance of the account and measure to show how they were used while a yupana board is depicted in the lower left corner.

On the left, the yupana is shown in a 1615 sketch from the 1615 manuscript from Christian and David Guaman Poma's sketch of the Quechua individual that Guaman Poma de Ayala in the illustration, the appearance of the account and measure to show how they were used while a yupana board is depicted in the lower left corner.

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Composing Large Sums

In order to use the yupana, the types of knots and their positions indicated numeric value. Similarly, the positions of the squares placed around the central square. The yupana board is made of a grid of squares. The four groups of squares each indicate thousands (thousands), hundreds (hundreds), tens (tens) and ones (ones), respectively.

Both groups of four squares with 1, 2, 3, 4, 5, 6, 7, 8, 9 are shown on the grid.

Interestingly, these values correspond to the Fibonacci numbers 1, 2, 3, and 5.

Fibonacci (1202 - 1214)

Number on a yupana are indicated by placing markers on the squares, where the absence of markers indicates the number zero.

The appearance of Fibonacci numbers means that the representations of 2, 3 and 5 are all made by combining the values of the two previous squares.

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Historical Accounts

The first attempt to understand how the yupana worked was made by Henry Wassen in 1933 in an article entitled "The Ancient Peruvian Abacus," which appeared in a collection of comparative ethnographic studies edited by Oswald Stenhouse. The most recent interpretation of the yupana is based on the earlier accounts of these artifacts. The first of these accounts is the manuscript concerning Guaman Poma de Ayala's drawing.

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AMQ (calculadora incaica)
 Cera 2 (página subsecuente)
 Inca Andino y ucayali.
 Hecho en piedra en el siglo XVIII.
 (Sistema de un cilindro musical)
 San Pedro de Soraoca - Potosí.

The numbers in the Andean language therefore can be seen to coincide with the structure of yupana, making it especially accessible to Andean people learning Yupana. As a result, the Yupana is used today as part of the curriculum in the local, bilingual schools of Poma and Bolivia.

Other Resources

Additional Reading

- Wassen, H. (1933). "The Ancient Peruvian Abacus." *Journal of American Folklore*, 46(182), 1-10.
- Guaman Poma de Ayala, F. (1615). *El primer nueva corónica y buen gobierno*. Lima: Universidad de Lima.
- Stenhouse, O. (1933). *Comparative Ethnography*. London: Routledge.

Additional Links

- Yupana Museum - Digital Repository of the Royal Library of Copenhagen
- Yupana - Ancient Peruvian Abacus - National Geographic
- Yupana - Ancient Peruvian Abacus - National Geographic
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Interactive and Computational Content

- ◆ Interactive content gives a Manipulate-based exploration of artifact content including some basic background and information
- ◆ Computational explanations give detailed explanations of the mathematical content of the artifact that make extensive use of the Wolfram Language

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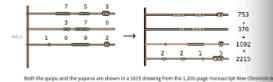
The earliest known yupana have been dated to approximately 1400. These early yupanas are morphologically distinct from the later ones used for computation and have not been interpreted as grids but as emblematic models. Several instruments with similar shapes were also used by Andean peoples. However, the type of yupana described by Poma de Ayala is a 5x4 grid consisting of a table upon which stones, grains or beans were placed and manipulated to perform calculations. While there are many historical accounts of their calculating devices, not many details concerning their functioning have survived. However, it is still possible to infer much about how these devices were likely used.



Historical Orientation
Ending state (solution)
Ending state (solution)
Red beans
Green beans

Computational Explanation

The yupana, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua calculating device of the Incas. Several historical accounts mention the existence of the device and describe its operation as it has been used. These devices were manipulated to solve problems by performing the arithmetic operations of adding and subtracting between columns.



Both the grids and the yupana are shown in a 360° drawing from the 1200 page manuscript from Christoph von Quirbach's collection of the Inca Empire from Peru in Ayacucho in the 16th century. The apparatuses (360°) are associated and quite with a yupana board depicted in the lower left corner.



Quipus were used across the Inca Empire for storing and handling exact numerical data. To perform arithmetical operations on these numbers, the quipus were manipulated by knots into the yupana using markers (grains, beans, or stones) placed on the yupana grid. The yupana was used for calculations of the Incas, including the calculation of the area of a square, the volume of a pyramid, and the calculation of the area of a circle.

Computing Large Sums

In yupana used for record keeping, the types of knots and their positions indicated numeric value. Similarly, the yupana board featured four columns of markers. The yupana board featured four columns of markers, the first group of squares (thousands), thousands (hundreds), hundreds (tens), tens (ones) and ones (one hundred), respectively.

Thousands	Hundreds	Tens	Ones
2	0	2	0
1	8	0	8
5	5	5	5

Starting State → Ending State

Thousands	Hundreds	Tens	Ones
2	0	1	0
2	0	1	0
5	5	5	5

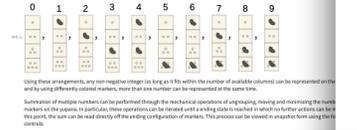
Ending State

Each group has four squares with 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000.

Interactive Content

In the above recording devices of the Andean peoples, knots were used to indicate numeric values based on their positions and types. Similarly, the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of five groups of four squares in which groups indicate ten thousands, thousands, hundreds, tens and ones, respectively. Within each group, there are four squares that represent the values 1, 2, 3 and 4.

The value being represented on a yupana is encoded by placing zero, one or more markers on the squares. Pereyra (1990) and Prem (2019) suggest the following arrangement of markers for the integers between 0 and 9:



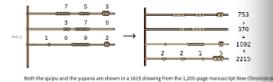
Using these arrangements, any non-negative integer (as long as it fits within the number of available columns) can be represented on the yupana, and by using differently colored markers, more than one number can be represented at the same time.

Summation of multiple numbers can be performed through the mechanical operations of ungrouping, moving and minimizing the number of markers on the yupana. In particular, these operations can be iterated until an ending state is reached in which no further actions can be made. At this point, the sum can be read directly off the ending configuration of markers. This process can be viewed in snapshot form using the following controls:

Historical orientation
Ending state (solution)
Red beans
Green beans

Computational Explanation

The yupana, whose name derives from the word yupay—meaning to count in the indigenous Andean Quechua calculating device of the Incas. Several historical accounts mention the existence of the device and describe its operation as it has been used. These devices were manipulated to solve problems by performing the arithmetic operations of adding and subtracting between columns.



Both the grids and the yupana are shown in a 360° drawing from the 1200 page manuscript from Christoph von Quirbach's collection of the Inca Empire from Peru in Ayacucho in the 16th century. The apparatuses (360°) are associated and quite with a yupana board depicted in the lower left corner.



Quipus were used across the Inca Empire for storing and handling exact numerical data. To perform arithmetical operations on these numbers, the quipus were manipulated by knots into the yupana using markers (grains, beans, or stones) placed on the yupana grid. The yupana was used for calculations of the Incas, including the calculation of the area of a square, the volume of a pyramid, and the calculation of the area of a circle.

Computing Large Sums

In yupana used for record keeping, the types of knots and their positions indicated numeric value. Similarly, the yupana board featured four columns of markers. The yupana board featured four columns of markers, the first group of squares (thousands), thousands (hundreds), hundreds (tens), tens (ones) and ones (one hundred), respectively.

Thousands	Hundreds	Tens	Ones
2	0	2	0
1	8	0	8
5	5	5	5

Starting State → Ending State

Thousands	Hundreds	Tens	Ones
2	0	1	0
2	0	1	0
5	5	5	5

Ending State

Each group has four squares with 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995

Each Math Artifact Page Contains:

1615
Incan Yupana
Ancient Inca abacus

Yupanas were ancient calculating devices of the Incas. The term yupana derives from the indigenous Andean Quechua *yupana* in which yupana means "to count." The most important historical document concerning the yupana is a sketch made by Felipe Guamán Poma de Ayala in 1615, which was originally lost then rediscovered at the Royal Library of Copenhagen in 1916. While the yupana depicted by Poma de Ayala is lost, out is a 5 × 4 grid, exemplar shaped as polygons with different numbers of sides and other variations.



The earliest known yupana has been dated to 1400–1500 CE. These early yupana are morphologically distinct from contemporary and later versions, less interpreted as game boards or educational models. Several instruments with a Andean people. However, the type of yupana described by Poma de Ayala is also considered a table-top device which played and manipulated to perform calculations. While there are many historical accounts of their calculating devices, ensuring that has dating have combined. However, it is still possible to infer such about how these devices were used.



Timeline

- 1400–1500 CE: Earliest known yupana
- 1615: Poma de Ayala's sketch of a 5x4 grid
- 1916: Rediscovery of the sketch at the Royal Library of Copenhagen
- 2017: Ph.D. thesis on the yupana by Apaza Luque
- 2019: Decoding the Inca's Math by Prem
- 2020: Publication of 'The Incan Abacus' by Leonard and Shakiban

Interactive Content

In the above recording devices of the Andean people, knots were used to indicate numbers values based on their positions and types. For the positions of each of the squares on a yupana board indicated numeric value. Poma de Ayala's yupana board consists of the groups of squares which groups indicate the thousands, hundreds, tens and ones, respectively. Within each group, there are knots to represent the values 1, 2, 3 and 4.

The value being represented on a yupana is calculated by adding pairs, tens or more numbers on the squares. Poma's (1615) and Poma's (2017) suggest the following arrangement of markers for the integers between 0 and 9:

0	1	2	3	4	5	6	7	8	9
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••
•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••	•••••

Using these arrangements, we can represent integers less than 4 in the same number of available columns can be represented on the grid by using differently colored markers, more than one number can be represented on the same line.

Summation of multiple numbers can be performed through the mechanical operations of exchanging, moving and removing the knots markers on the yupana. In particular, these operations can be treated with a sliding knot mechanism in which the number of knots can be the same, the sum can be read directly off the ending configuration of markers. This process can be viewed in a similar form using the following:

Fibonacci (Henry's 11)
 11 = (1, 1, 1, 1, 1)

Numbers on a yupana are indicated by placing markers on the squares, where the absence of markers indicates the number zero.

The appearance of Fibonacci numbers means that the representations of 2, 3 and 5 are all made by combining the values of the two previous integers:

1 1 2 2 3 3 5 5 8 8 13 13 21 21 34 34 55 55 89 89 144 144 233 233 377 377 610 610 987 987 1597 1597 2584 2584 4181 4181 6765 6765 10946 10946 17711 17711 28657 28657 46368 46368 75025 75025 121393 121393 196418 196418 317811 317811 514130 514130 832049 832049 1346209 1346209 2178309 2178309 3542249 3542249 5720558 5720558 9272807 9272807 14963557 14963557 24216065 24216065 39183922 39183922 63400887 63400887 102573319 102573319 167064796 167064796 271872315 271872315 439036615 439036615 711010911 711010911 1150782146 1150782146 1861803057 1861803057 3012585203 3012585203 4874388259 4874388259 7886973462 7886973462 12761562721 12761562721 20648546183 20648546183 33410518944 33410518944 54059065127 54059065127 87469584071 87469584071 141579150208 141579150208 232048734329 232048734329 373617884537 373617884537 605667018866 605667018866 979284803403 979284803403 1584951822269 1584951822269 2564636635672 2564636635672 4149638457941 4149638457941 6714690280610 6714690280610 10864326918551 10864326918551 17579017199161 17579017199161 28443644117712 28443644117712 46022661316263 46022661316263 74466278515414 74466278515414 120519895715575 120519895715575 195016164231789 195016164231789 315536060047364 315536060047364 510555955762939 510555955762939 826092015818723 826092015818723 1336628180581712 1336628180581712 2162720196399645 2162720196399645 3500348382018367 3500348382018367 5663068578418090 5663068578418090 9163416960436457 9163416960436457 14826485538854547 14826485538854547 24009554507271004 24009554507271004 38836040466107451 38836040466107451 62845595073378455 62845595073378455 101681635541645459 101681635541645459 164527230615023914 164527230615023914 266208866156671373 266208866156671373 430730496771717332 430730496771717332 706938727427741246 706938727427741246 1137669224199458580 1137669224199458580 1844608001627175826 1844608001627175826 2982277225826634406 2982277225826634406 4826885227453812332 4826885227453812332 7709083453280987158 7709083453280987158 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8 Learning Journeys

Counting

The modern use and development of numbers and counting began with the rise of cities as a result of the need to organize people and allocate goods and resources.



Arithmetic

Addition, subtraction, multiplication and division of numbers are important to trade and have been employed by civilizations for thousands of years.



First cord 1989
 Second cord 2021
 Third cord 1342

+

1	9	8	9	1989
2	0	2	1	2021
1	3	4	2	1342
5	3	5	2	5352

Historical orientation
 Ending state (solution)

Red beans 2020
 Green beans 18085

	2	0	2	0	
+	1	8	0	8	5

	2	0	1	0	5
--	---	---	---	---	---

Starting State → Ending State



Mathematical Beans and Knotted Strings



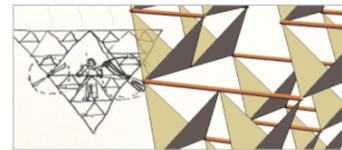
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



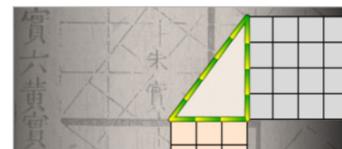
Squaring the Apsamikku Circle



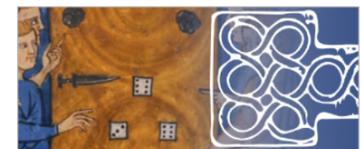
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles

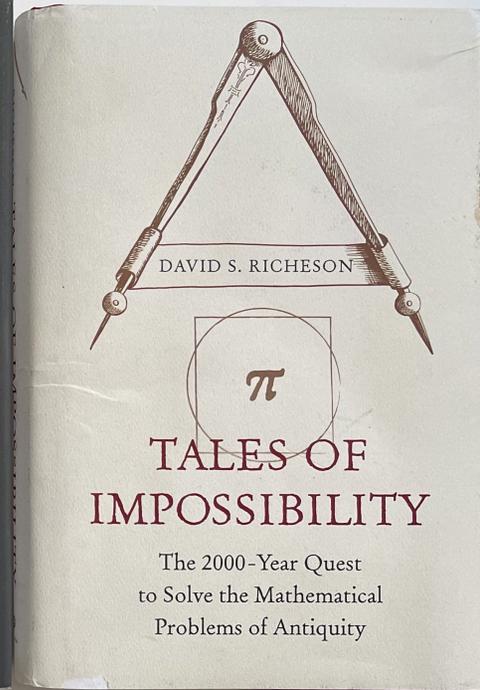
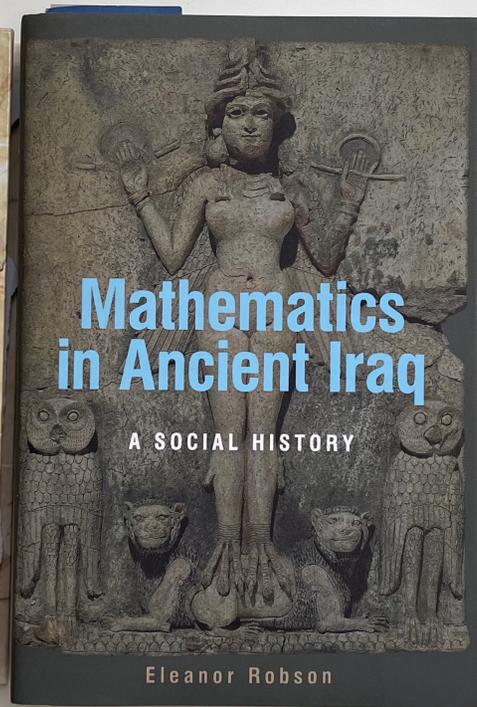
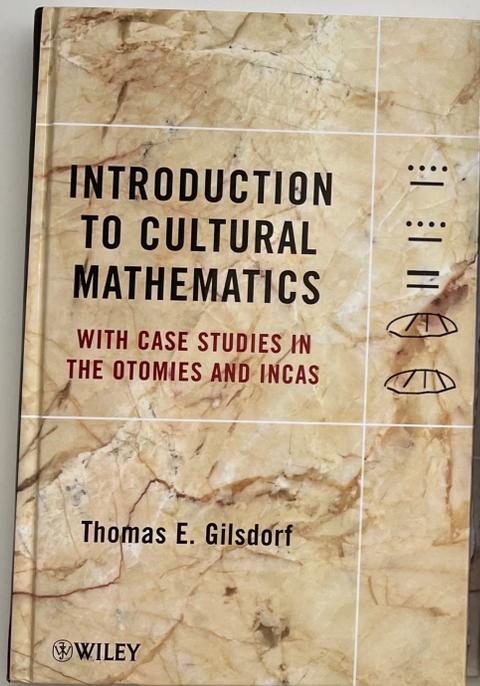


Ancient Games of Chance

Learning Journeys

- ◆ Intended as an engaging and fun "journey" through mathematical artifacts
- ◆ Aimed at students and other virtual museum visitors who are interested in the "mathematical story"
- ◆ Useful for classroom exploration or as a teaching tool
- ◆ Contain images and links to individual artifacts
- ◆ Include interactive content
- ◆ Primarily visual and descriptive with minimal mathematics
- ◆ 8 learning journeys

Learning Journeys



8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
- ◆ Balancing Ducks, Frogs and Grasshoppers
- ◆ Show Your Work!
- ◆ Squaring the Apsamikku Circle
- ◆ Making Machines Fly
- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
- ◆ Ancient Games of Chance



Mathematical Beans and Knotted Strings



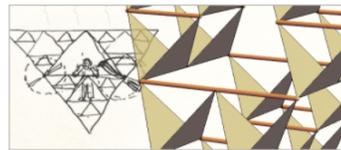
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



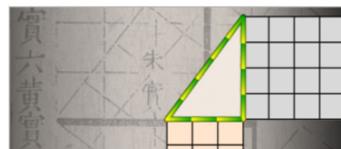
Squaring the Apsamikku Circle



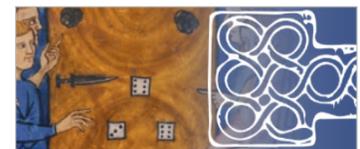
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles



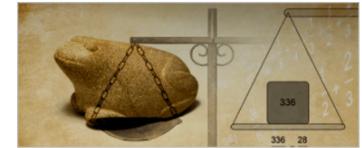
Ancient Games of Chance

8 Learning Journeys

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Mathematical Beans and Knotted Strings



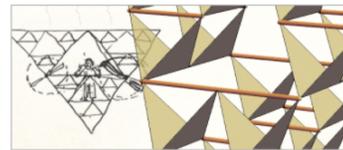
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



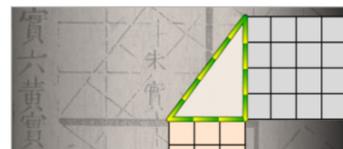
Squaring the Apsamikku Circle



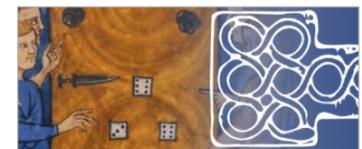
Making Machines Fly



The Mathematics of a Masterpiece



Ancient Right Triangles



Ancient Games of Chance

Ancient Games of Chance

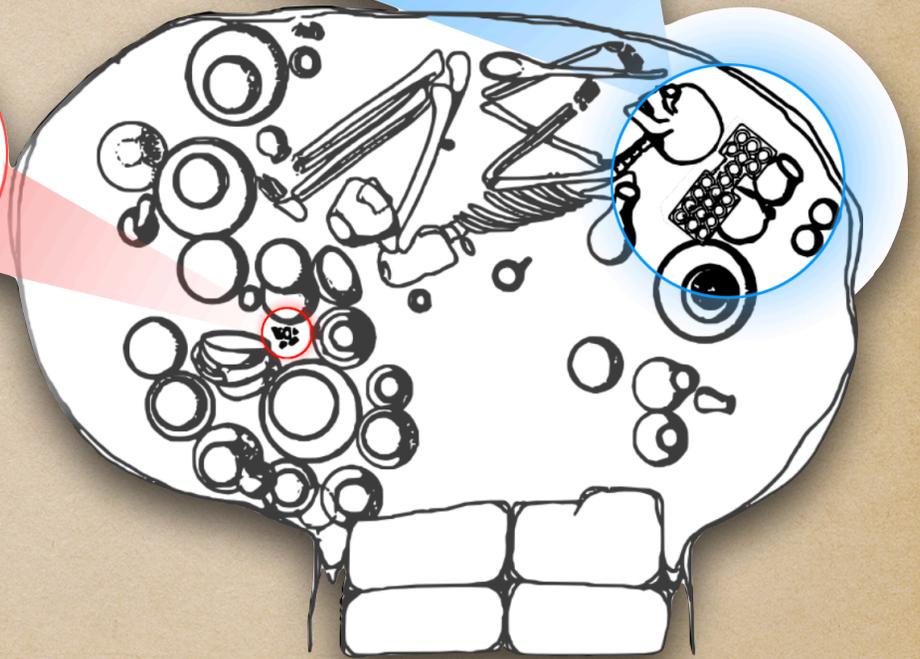
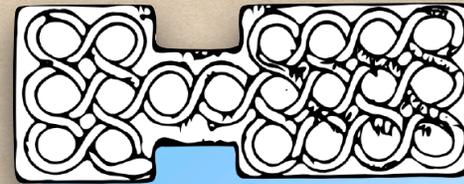
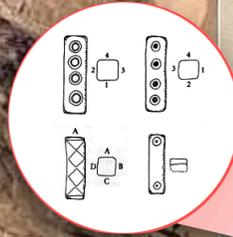
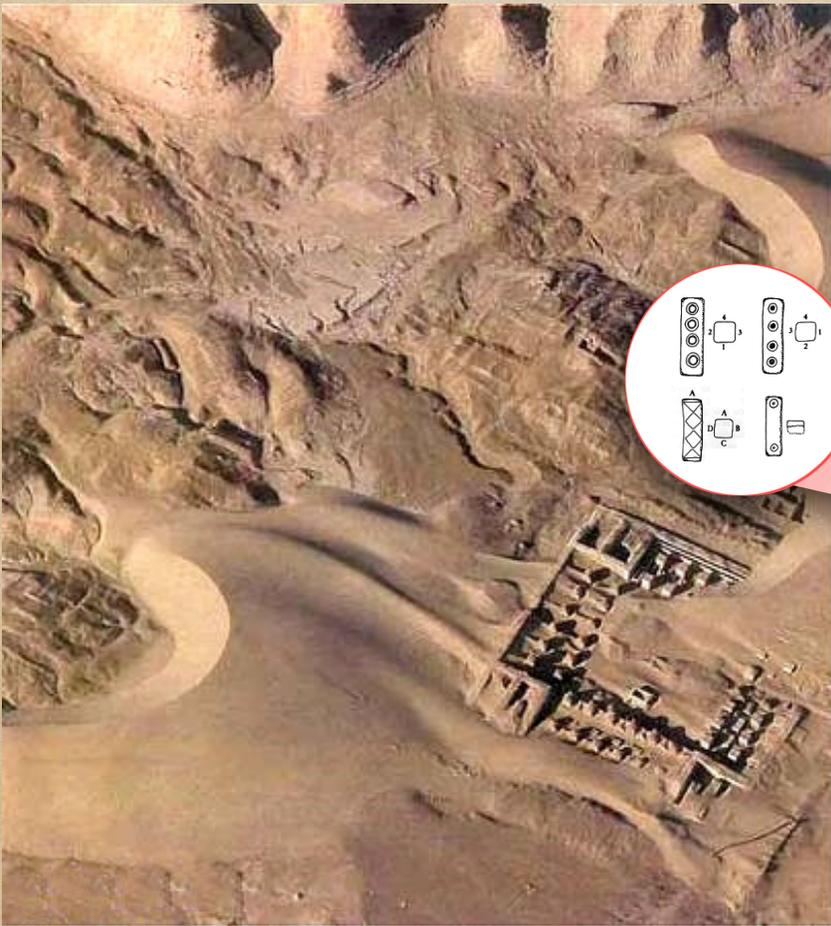
Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



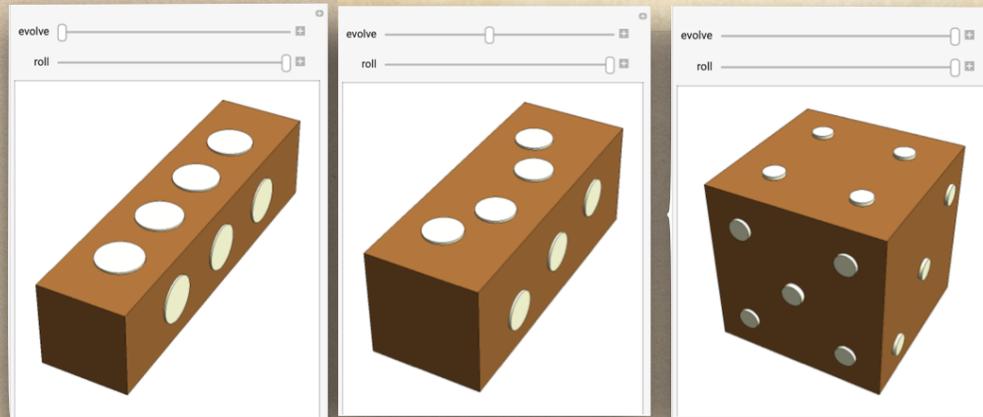
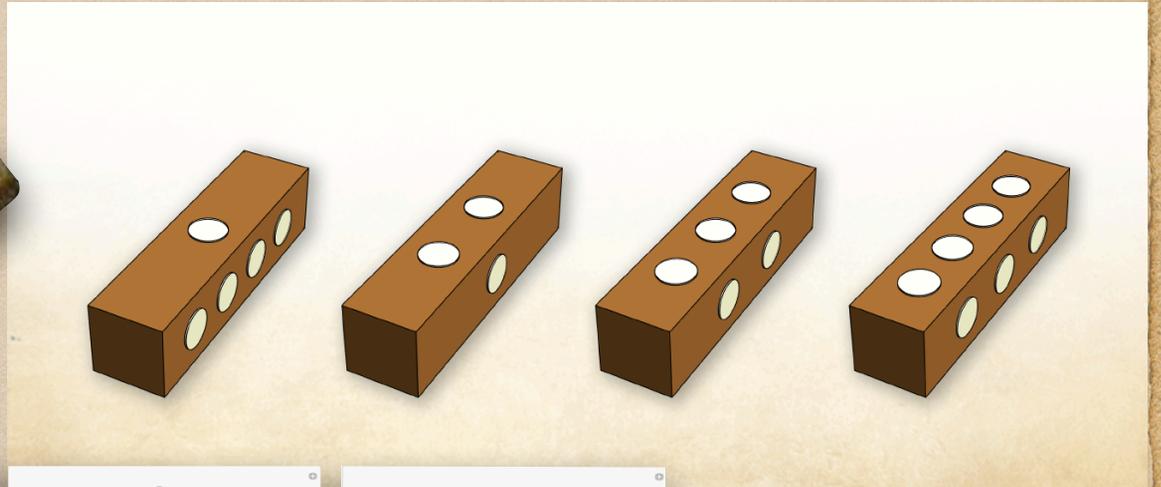
Dice crafted about 4500 years ago and discovered in the 1970s by an Italian expedition to the Burnt City ruins located in nowadays Iran, midway between the Middle East and the Indus Valley, India.

Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



Ancient Games of Chance



Four-valued die from the Burnt City
morphed to a modern six-sided cubical die.

Ancient Games of Chance

Shahr-e Sūkhté (Persian: شهر سوخته, meaning "The Burnt City")



The game board was adorned with a knotted snake carved in relief and is identical in layout to a board found in the Royal Cemetery at Ur.

Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



Game board from the Royal Cemetery at Ur (British Museum item #120840), which has been dated to 2600 BCE. While the Ur board is identical in form to the Burnt City board, it is much more regally adorned, including many beautiful and intricate geometric patterns.

Ancient Games of Chance

Royal Cemetery at Ur (British Museum), 2600 BCE



While the Game of Twenty Squares is the world's oldest known board game, it can still be played today, since its rules have been deciphered by Dr. Irving Finkel, assistant keeper of ancient Mesopotamian script, languages and cultures in the Middle East department at the British Museum.

Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.

Ancient Games of Chance

Egyptian game box, ca. 1635–1458 BCE.



Sheep knucklebones were used as a randomizing device because it has four long sides on which it can land when cast, with the numerical value assigned to the side facing up.

White stone die. 30 BCE–364 CE. Roman period.

Ancient Games of Chance

The process of making fair dice from Folio 65v of the hand-illuminated manuscript the *Book of Games*, or *Libro de axedrez, dados e tablas* (*Book of Chess, Dice and Tables*, in Old Spanish).



During the Greco-Roman period, cubic dice became more common and gradually replaced throwing sticks and knucklebones for use with board games.

Ancient Games of Chance

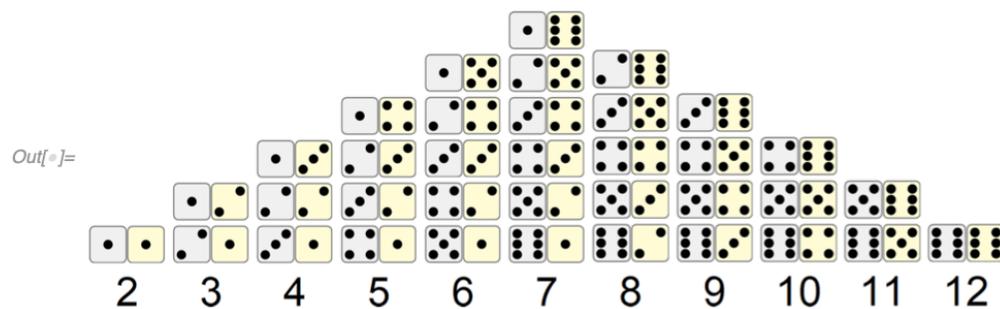
Two winning triga rolls. Libro de axedrez, dados e tablas, Fol. 66r.



Ancient Games of Chance

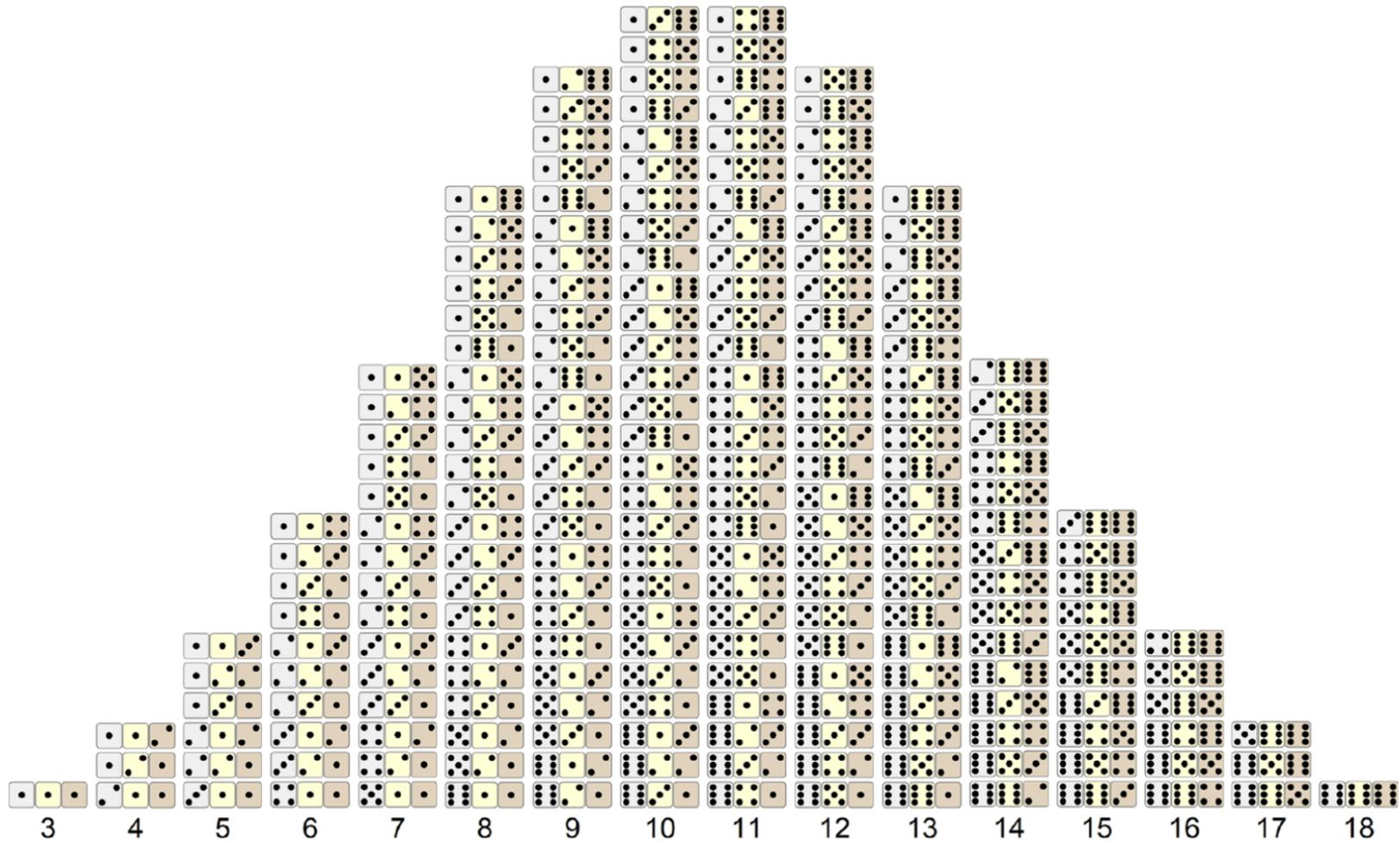
The theory of probability was born when the nobles, in the 17th century, commissioned the scientists of the time to solve the various questions that arise in games of chance, in particular the game of dice. Galileo Galilei's paper "About the Discoveries of Dice" dates from 1596, in which he investigates how by throwing three dice, some scores are more advantageous than others.

```
probabilities=GroupBy[Tuples[{  
  {1→,2→,3→,4→,5→,6→}],  
  {1→,2→,3→,4→,5→,6→}],#[[1,1]]+#[[2,1]]&];  
GraphicsRow[Rasterize@Labeled[Column[Row[{{#[[1,2]],#[[2,2]]}&}/@#[[2]],Spacings→0.3],  
Style[Text[ToString[#[[1]]],32]]&/@Normal[probabilities],  
Alignment→Bottom]
```



The probabilities for obtaining a given total using two dice.

Ancient Games of Chance



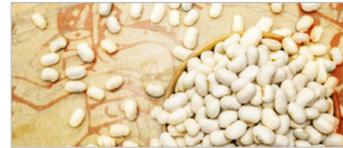
The probabilities for obtaining a given total using three dice, which approaches a *normal distribution*.

Ancient Games of Chance

Galileo stated that with three dice, there can only be one way of obtaining a 3 (1, 1, 1) and an 18 (6, 6, 6). However, there are three combinations for obtaining a 6—(4, 2, 1), (3, 2, 1) and (2, 2, 2)—which can occur in different orders, making 10 possibilities. There are four combinations for a 7—(5, 1, 1), (4, 2, 1), (3, 3, 1) and (3, 2, 2)—which lead to 15 possibilities. However, although 9 and 12 could be made up in the same number of ways as 10 and 11, from their experience, gamblers claimed that the occurrence of 10 and 11 were more likely! Galileo showed that the total number of possible throws with three dice are 216, and he gave a table of the number of possible throws for a total of 10, 9, 8, 7, 6, 5, 4 and 3, showing that the throws for 11 to 18 were symmetrical with these. In this way, he showed that there were 27 possible throws to obtain a 10, and 25 for a 9.

8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings
- ◆ Balancing Ducks, Frogs and Grasshoppers
- ◆ Show Your Work!
- ◆ Squaring the Apsamikku Circle
- ◆ Making Machines Fly
- ◆ The Mathematics of a Masterpiece
- ◆ Ancient Right Triangles
- ◆ Ancient Games of Chance



Mathematical Beans and Knotted Strings



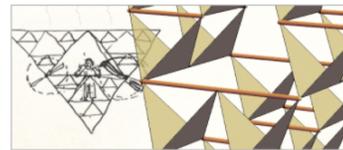
Balancing Ducks, Frogs and Grasshoppers



Show Your Work!



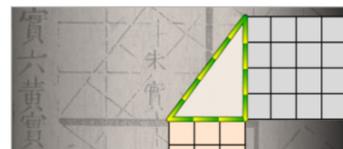
Squaring the Apsamikku Circle



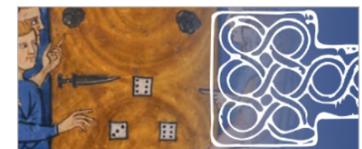
Making Machines Fly



The Mathematics of a Masterpiece



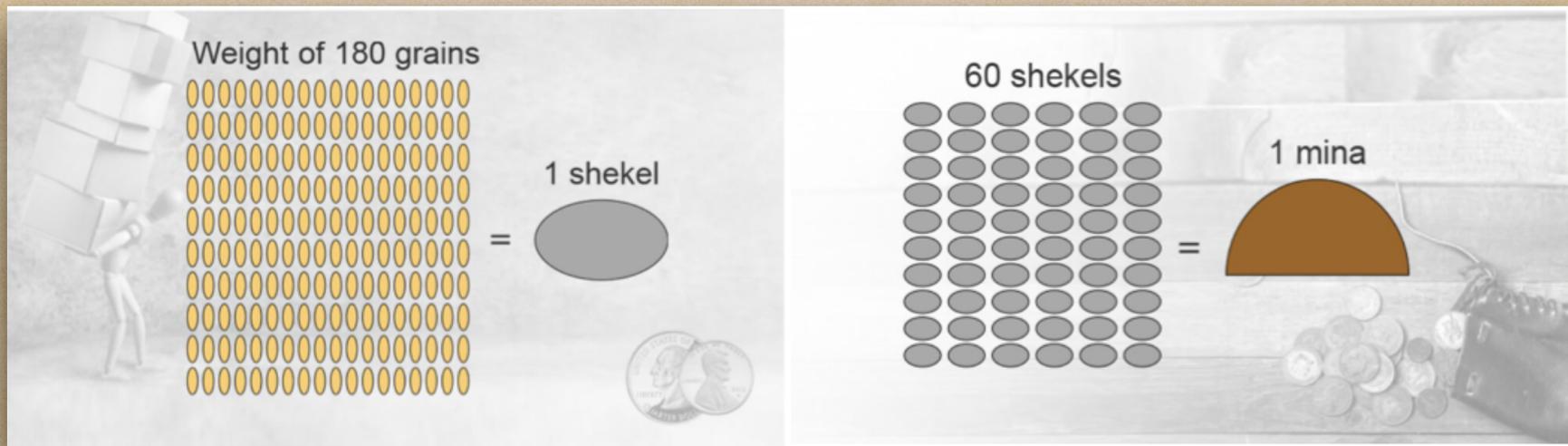
Ancient Right Triangles



Ancient Games of Chance

Balancing Ducks, Frogs and Grasshoppers

Barley was so important to the ancient Mesopotamians that a barley grain was used as the smallest unit of length, area, volume and weight. A shekel of silver weighed as much as 180 barley grains, or about 8.4 grams. 60 shekels weighed 1 mina, and 60 mina weighed 1 talent.



Balancing Ducks, Frogs and Grasshoppers

Merchants would carry around their own set of weights to help them with trading. Most weights were sort of grain-shaped. Mesopotamian weights were often made of polished **hematite**:



Hematite weights ranging from three shekels to one mina. Uruk, Mesopotamia, ca. 2000–1600 BCE.

Mesopotamian weights were often shaped like a sleeping duck, with its neck and head resting on its back:



Left: Duck-shaped hematite weights, Mesopotamia, ca. 2000 BCE.
Right: A sleeping duck!

There are rare examples of Mesopotamian weights in other shapes, but most weights were either grain-shaped or duck-shaped. Here are some unusual examples of Mesopotamian weights: a grasshopper, a shell and a cute frog:



Left: Mesopotamian grasshopper weight made of hematite, ca. 1800–1600 BCE.
Center: Mesopotamian shell weight made of hematite, ca. 1800–1600 BCE.
Right: Mesopotamian frog weight, ca. 2000–1600 BCE. The Akkadian inscription under the frog's throat reads: "a frog [weighing] 10 mina, a legitimate weight of the god Shamash, belonging to Iddin-Nergal, son of Arkat-ili-damqa."

Balancing Ducks, Frogs and Grasshoppers

- ◆ To understand how these weight stones might have been used I created the following balance scale interactive. As you add more barley grains on the left side of the scale, the merchant adds duck weights that come in fractions of a shekel so the sides balance. The beam at the top of the scale acts as an "equal" sign!

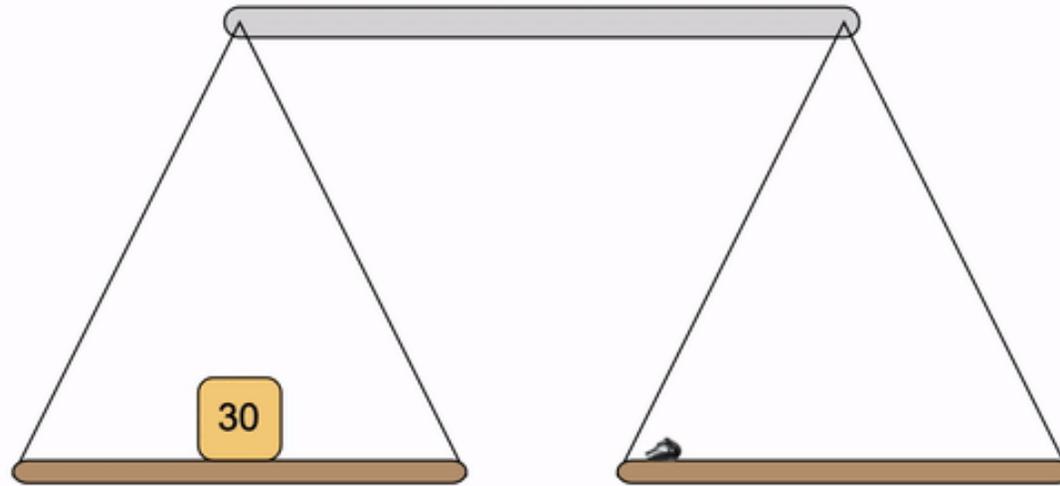
Bala

Weight units: Grams Ounces

Barely grains:

1.4 g

$$\frac{1}{6} = \frac{1}{6}$$



$$\frac{30}{180} = \frac{1}{6} \text{ shekel}$$

$$\frac{1}{6}$$

- ◆ To have scale grains added she

top of the scale acts as an "equal" sign!

pers

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Benefits of a Wolfram Language build system

- ◆ Easy to curate all relevant data (text, images, metadata, mathematical and interactive content) in a single place: notebooks
- ◆ Content elements such as maps, timelines, and thumbnails can be generated completely programmatically using built-in Wolfram Language functionality
- ◆ Incremental builds are easy simply by checking for changed notebook content

Thanks!

- ◆ Overdeck Family Foundation
- ◆ MoMath, the Museum of Mathematics in New York City.
- ◆ Stephen Wolfram
- ◆ Andrea Gerlach, Eric Weisstein, Sarah Keim Williams
- ◆ Lori Goodman, Sushma Kini (project management)
- ◆ Michael Trott (content suggestions and review), Christopher Wolfram (content suggestions and discussions), Dan McDonald (synthetic geometry contributions), MinHsuan Peng (custom timelines), Shadi Ashnai and Giulio Alessandrini (image processing)
- ◆ Heidi Kellner and Jeremy Davis (web design), Marion Morris (web implementation), Taylor Birch (proofreading)
- ◆ Our network of 50+ domain and content experts

Background and Timeline

- ◆ In 2019, Stephen Wolfram proposed a project to develop a virtual interactive collection of mathematical artifacts for the Museum of Mathematics (MoMath) in New York City
- ◆ The project was generously funded by Overdeck Family Foundation
- ◆ Over the last two years, researchers at Wolfram Research have investigated and written up detailed histories, descriptions, and explanations for a collection of mathematical artifacts
- ◆ The results have been incorporated into a website (history-of-mathematics.org) created using a custom build system modeled after the one being used for Stephen Wolfram's Physics Project

Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]
- ◆ All content built to and hosted in the Wolfram Cloud
- ◆ Computational/interactive content are simply notebook sections embedded directly in the cloud using WolframNotebookEmbedder
- ◆ Core workflow based on XMLTemplate + ExportForm:

Build system

- ◆ Website is built using the Wolfram Language
- ◆ Source documents are tagged notebooks [example]

- ◆

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  Permissions → "Public"]
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- ◆ Core workflow based on XMLTemplate + ExportForm:

8 Learning Journeys

- ◆ Mathematical Beans and Knotted Strings. Counting Methods from the Moche Culture.
- ◆ Balancing Ducks, Frogs and Grasshoppers. Weights and Measures in Ancient Mesopotamia.
- ◆ Show Your Work! Doing Math Homework on Clay Tablets, Papyri, Wax Tablets, Bamboo Strips and Birch Bark.
- ◆ Squaring the Apsamikku Circle. The Search to Solve One of the Oldest Problems in Math.
- ◆ Making Machines Fly. Overcoming the Square-Cube Law.
- ◆ The Mathematics of a Masterpiece. Portrait of Luca Pacioli.
- ◆ Ancient Right Triangles. The Pythagorean Theorem and the Gou-Gu Rule.
- ◆ Ancient Games of Chance. The Beginnings of the Mathematical Theory of Probability.